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VISUALIZATION OF THE EIGENVALUE PROBLEMS OF THE LAPLACIAN FOR EMBEDDED SURFACES AND ITS APPLICATIONS

by

Masaki Jumonji & Hajime Urakawa

Abstract. — In this paper, we give numerical computations and computer visualizations of the various eigenvalue problems of the Laplacian for plane domains, embedded closed surfaces and their enclosed domains in the three dimensional Euclidean Space, and show their applications.

***Résumé* (Visualisation des problèmes de valeurs propres pour le laplacien des surfaces plongées et ses applications)**

Nous donnons des calculs numériques et des représentations par ordinateur de différents problèmes associés aux valeurs propres du Laplacien pour des domaines plans, des surfaces fermées plongées et les domaines qu'elles entourent dans un espace euclidien de dimension 3. Et nous montrons leurs applications.

1. Introduction.

Since R. Courant [7], numerical computations on the solutions of the partial differential equations have widely been studied and applied to various industries. They are almost impossible for us to survey all of them and to get even a bird's eye view (see [1]). In this paper, we concentrate ourselves the eigenvalue problems of the Laplacian and related topics. We first prepare the materials for the setting on the eigenvalue problems of the Laplacian on compact Riemannian minfolds with or without boundary. Let (M, g) be a d-dimensional compact Riemannian manifold with or without

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boundary ∂M. Let Δ be the Laplacian of (M,g) acting on the space $C^\infty(M)$ of smooth functions on M. That is, Δ is given by

$$\Delta f = -\frac{1}{\sqrt{\det(g_{k\ell})}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(\sqrt{\det(g_{k\ell})}\, g^{ij}\, \frac{\partial f}{\partial x^j}\right), \quad f \in C^\infty(M),$$

where $(x_1,\dots,x_d)$ is a local coordinate on M, $(g^{ij}) = (g_{kl})^{-1}$, and (g_{kl}) are the components of g with respect to the coordinate. If $\partial M = \varnothing$, we consider the free boundary eigenvalue problem of the Laplacian Δ

$$\Delta\varphi = \lambda\varphi \quad (\text{on} \quad M), \tag{1.1}$$

and if $\partial M \neq \varnothing$, the Dirichlet eigenvalue problem (the Neumann eigenvalue problem)

$$\begin{cases} \Delta\varphi &= \lambda\varphi \quad \text{on } M, \\ \varphi &= 0 \quad \text{on } \partial M, \end{cases} \tag{1.2}$$

or

$$\begin{cases} \Delta\varphi &= \lambda\varphi \quad \text{on } M, \\ \frac{\partial\varphi}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial M, \end{cases} \tag{1.3}$$

where $\mathbf{n}$ is the inward unit normal vector field along ∂M. Then, they have the discrete spectra, which we denote by

$$\mathrm{Spec}(M,g) = \{0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty\}$$

for (1.1),

$$\mathrm{Spec}_D(M,g) = \{(0 <)\lambda_1^D \leq \lambda_2^D \leq \cdots \leq \lambda_k^D \leq \cdots \to \infty\}$$

for (1.2), and

$$\mathrm{Spec}_N(M,g) = \{0 = \lambda_1^N < \lambda_2^N \leq \cdots \leq \lambda_k^N \leq \cdots \to \infty\}$$

for (1.3), and the corresponding eigenfunctions for (1.1), (1.2) and (1.3) by φ_k, φ_k^D, and φ_k^N, $(k = 1, 2, \cdots)$, respectively. We take for the eigenfunctions φ_k, φ_k^D, and φ_k^N to be orthonormal with respect to the inner product $(\ ,\ )$ which is given by

$$(f,h) = \int_M f(x)h(x)v_g,$$

for f and h in $C^\infty(M)$. Here v_g is the volume element defined by $v_g = \sqrt{\det(g_{ij})}dx_1 \wedge \cdots \wedge dx_d$.

So far, the only known examples for the spectrum of the free boundary problem (1.1) in the case $\partial M = \varnothing$, are the Riemannian symmetric spaces including the unit sphere. In the case of the unit sphere (S^d, g_0), the eigenvalues are $l(l+d-1)$, $(l = 0, 1, \cdots)$, their multiplicities m_l are given by $m_l = \frac{(l+d-2)!}{(d-1)!l!}(2l+d-1)$, and the corresponding eigenfunctions are the restriction of the k-th harmonic polynomials on $\mathbb{R}^{d+1}$ to S^d. For all the embedded compact surfaces except the unit sphere, the explicit eigenvalues and eigenfunctions are completely unknown.

In this paper, we consider (1.1), (1.2) or (1.3) for an arbitrarily embedded compact surface M without (or with) boundary in the three dimensional Euclidean space $\mathbf{R}^3$, and the induced Riemannian metric g from the standard Riemannian metric g_0, that is, $g = \iota^* g_0$, where ι is the inclusion of M into $\mathbf{R}^3$. We will show that the finite element method works well to compute numerically the eigenvalues and the eigenfunctions of (M, g) and as its applications, we will show the computer graphics on the eigenvalues and eigenfunctions, and the solutions of the heat and wave equations on the embedded surfaces in the three dimensional Euclidean space. The construction of this paper is as follows:

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2. The FEM for the eugenvalue problems of the Laplacian.

In this section, we treat with the finite element method initiated by R. Courant, to compute numerically by computers the eigenvalues and eigenfunctions of the Laplacian for the problems (1.1), (1.2) or (1.3), respectively. Let (M, g) be an embedded compact surface as in the intoduction. Let us denote the standard coordinate of $\mathbf{R}^3$ by (x, y, z). Let us take a triangulation $\Xi = \{e_\mu\}$ of M. That is, let us take a set of points in M, $\{P_1, \ldots, P_m\}$, ($\{P_1, \ldots, P_\ell, P_{\ell+1}, \ldots, P_m\}$, the set of the one in M such that $\{P_{\ell+1}, \ldots, P_m\}$ is the set of points belonging to ∂M if $\partial M \neq \varnothing$) and let $\Xi = \{e_\mu\}$ be the set of triangles in $\mathbf{R}^3$ given by straight segments joining the above two points. Let $G(\Xi) = \bigcup_\mu e_\mu$. If we take enough many points $\{P_1, \ldots, P_m\}$ in M, $G(\Xi)$ approximates M well. Let ψ_i $(i = 1, \ldots, m)$ be the basic functions on $G(\Xi)$ associated to Ξ, i.e.,

1. $\psi_i(x,y,z)$ is at most linear on each e_μ, i.e.,

$$\psi_i(x,y,z) = a_i^\mu x + b_i^\mu y + c_i^\mu z \quad \text{if } (x,y,z) \in e_\mu,$$

for some constants a_i^μ, b_i^μ and c_i^μ for e_μ, and

2. $\psi_i(P_j) = \delta_{ij}$.

Then, we obtain the two $m \times m$-matrices for (1.1) if $\partial M = \varnothing$ (or the two $\ell \times \ell$-matrices for (1.2) if $\partial M \neq \varnothing$) $K(\Xi) = (K_{ij}(\Xi))$ and $M(\Xi) = (M_{ij}(\Xi))$, called the *stiffness matrix* and the *mass matrix* given by

$$\begin{cases} K_{ij}(\Xi) & = \int_{G(\Xi)} \langle \nabla\psi_i, \nabla\psi_j \rangle_{g(\Xi)} v_{g(\Xi)}, \\ M_{ij}(\Xi) & = \int_{G(\Xi)} \psi_i \psi_j v_{g(\Xi)}, \end{cases}$$

where $\langle\ ,\ \rangle_{g(\Xi)}$ is the inner product with respect to the continuous Riemannian metric $g(\Xi) = \iota^* g_0$ on $G(\Xi)$ induced from the inclusion $\iota : G(\Xi) \subset \mathbf{R}^3$. Then, we have

Definition 2.1. — (1) *Assume that $\partial M = \varnothing$. For each vector $\mathbf{u} = {}^t(u_1, \ldots, u_m)$ in $\mathbf{R}^m$, we define the piecewise smooth function $\hat{\mathbf{u}}$ on $G(\Xi)$ by*

$$\hat{\mathbf{u}}(x,y,z) = \sum_{s=1}^{m} u_s \psi_s(x,y,z), \quad (x,y,z) \in G(\Xi). \tag{2.1}$$

(2) *Assume that $\partial M \neq \varnothing$. For every $\mathbf{u} = {}^t(u_1, \ldots, u_\ell) \in \mathbf{R}^\ell$, we also define*

$$\hat{\mathbf{u}}(x,y,z) = \sum_{s=1}^{\ell} u_s \psi_s(x,y,z), \quad (x,y,z) \in G(\Xi). \tag{2.2}$$

Then, this function $\hat{\mathbf{u}}$ vanishes at the boundary $\partial G(\Xi)$.

Definition 2.2. — (1) *Assume that $\partial M = \varnothing$. Let us consider the eigenvalue problem for the $m \times m$-matrices $K(\Xi)$ and $M(\Xi)$,*

$$K(\Xi)\mathbf{u} = \nu M(\Xi)\mathbf{u}, \quad \mathbf{u} \in \mathbf{R}^m \tag{2.3}$$

and we denote by $\{\nu_1(\Xi), \nu_2(\Xi), \ldots, \nu_m(\Xi)\}$, the eigenvalues of (2.3) counted with their multiplicities, and by $\{\mathbf{u}_1(\Xi), \mathbf{u}_2(\Xi), \ldots, \mathbf{u}_m(\Xi)\}$, the corresponding eigenvectors of (2.3), respectively.

(2) *Assume that $\partial M \neq \varnothing$. For the $\ell \times \ell$-matrices $K(\Xi)$ and $M(\Xi)$, we consider the similar eigenvalue problem*

$$K(\Xi)\mathbf{u} = \nu M(\Xi)\mathbf{u}, \quad \mathbf{u} \in \mathbf{R}^\ell. \tag{2.4}$$

Denote by $\{\nu_1^D(\Xi), \nu_2^D(\Xi), \cdots, \nu_\ell^D(\Xi)\}$ and $\{\mathbf{u}_1^D(\Xi), \mathbf{u}_2^D(\Xi), \ldots, \mathbf{u}_\ell^D(\Xi)\}$, for the eigenvalue problem (2.4), respectively.

Then, it is well known (cf. [2], [3], [9], [11], [13]) the following: