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Tristan Rivière

## ANALYTIC ASPECTS OF PROBLEMS IN RIEMANNIAN GEOMETRY: ELLIPTIC PDES, SOLITONS AND COMPUTER IMAGING

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# THE ROLE OF INTEGRABILITY BY COMPENSATION IN CONFORMAL GEOMETRIC ANALYSIS

*by*

Tristan Rivière

***Abstract.*** — We study the solutions to the Euler-Lagrange equations that derive from conformally invariant Lagrangians in 2-dimensions. Particular examples come from the constant mean curvature equation for a surface, the harmonic map equation, and more general elliptic systems of quadratic growth. We are concerned with regularity issues, conservation laws and the lack of compactness of the space of solutions below a certain energy level. In particular we explain how the answer to the latter question is related to so-called integrability by compensation.

***Résumé*** **(Le rôle d'intégrabilité par compensation dans l'analyse géométrique conforme)**

Nous étudions les solutions des équations d'Euler-Lagrange associées à des Lagrangiens invariants conformes en dimension 2. En particulier, des exemples de ce type de solutions proviennent de l'équation de la courbure moyenne d'une surface, de l'équation des applications harmoniques et plus généralement de systèmes elliptiques à croissance quadratique. Nous nous intéressons aux résultats de régularité, aux lois de conservation et au manque de compacité de l'espace des solutions en dessous d'un certain niveau d'énergie. En particulier, nous expliquons comment la solution de la question précédente est reliée à l'intégrabilité par compensation.

## 1. Wente's inequalities and integrability by compensation

The *Integrability by Compensation* is an improvement in the a-priori rate of integrability of a function due to special cancellation, compensation, phenomena. It was probably first discovered by Henry C. Wente in [51] in his work on constant mean curvature surfaces.

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Given two functions $a$ and $b$ in the Sobolev Space $W^{1,2}(\omega, \mathbb{R})$ of $L^2$ functions on a domain $\omega$ of $\mathbb{R}^2$ whose distributional derivatives are also in $L^2$, the jacobian function

$$\frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} \tag{1.1}$$

is a-priori only in $L^1(\omega)$. The observation made by H. C. Wente was that the convolution of this function together with the Green Kernel $\log|x|$ of the Laplacian is in the Sobolev Space $W^{1,2}_{loc}(D^2)$ and also in $L^\infty_{loc}(D^2)$. This realizes an improvement of the a-priori integrability properties given by classical singular integral theory. Indeed the convolution of an $L^1$ function together with the Green Kernel $\log|x|$ is a-priori only in the space of Bounded Mean Oscillations BMO and that the derivatives of such a convolution are a-priori only in the weak $L^2$ space (or Marcinkiewicz space) $L^{2,\infty}$ (see [41]). Recall the characterization of $L^{2,\infty}$ measurable function on a domain $\Omega$:

$$\|f\|_{L^{2,\infty}(\omega)} \simeq \sup_{\lambda>0} \ \lambda\ |\{x\in\omega\ ;\ |f(x)|\ge\lambda\}|^{\frac{1}{2}}\ .$$

This improvement of intergrability, or regularity, is due to the special algebraic structure of the quadratic nonlinearity (1.1) which is of jacobian type.

Later on, in [42], Luc Tartar wrote a new proof of Wente's result using an argument which permitted to improve the gain of integrability obtained by Wente. Tartar indeed established that the Fourier transform of the convolution between the jacobian (1.1) and the Green function $\log|x|$ was in a strictly smaller space than $L^2$: the Lorentz Space $L^{2,1}$ dual to the weak $L^2$ space $L^{2,\infty}$. We recall now a characterization of the $L^{2,1}$ space: a measurable function $f$ on $\Omega$ is in the Lorentz space $L^{2,1}$ if and only if

$$\int_0^{+\infty} |\{x\in\Omega\ ;\ |f(x)|\ge\lambda\}|^{\frac{1}{2}}\ d\lambda < +\infty$$

A systematic presentation of Lorentz spaces can be found in [44]. In [19] section 3.4, Frédéric Hélein presents another argument by Luc Tartar, based on the use of an interpolation result of Jacques-Louis Lions for bilinear operators, which shows that derivatives of the convolution between the Green Kernel $\log|x|$ and the jacobian (1.1) are themselves in the Lorentz space $L^{2,1}$. This permits in particular to recover the full result of Wente since functions on a 2-dimensional domain whose gradients are locally in $L^{2,1}$ are continuous.

In [34], Stefan Müller proved, under the additional assumption that the jacobian (1.1) has a sign, that, still assuming that $a$ and $b$ are in $W^{1,2}$, this jacobian is in a smaller space than $L^1$: the Orlicz space $L^1\log L^1$. As a consequence, using the classical theory of Calderon Zygmund operators, see [41], one then obtain that the convolution between the Green Kernel $\log|x|$ and the jacobian (1.1) is locally in the Sobolev Space $W^{2,1}$ which permits in particular, using Lorentz-Sobolev embeddings-see [43] and [44], to recover Wente's and Tartar's results under this sign assumption.

Later on, Ronald Coifman, Pierre-Louis Lions, Yves Meyer and Stephen Semmes were able to drop the sign assumption made by Müller and proved that, under the assumption that $\nabla a$ and $\nabla b$ are in $L^2$ only, the jacobian (1.1) is in the local Hardy space $\mathcal{H}^1_{loc}$. Recall that, for positive functions, the local Hardy space $\mathcal{H}^1_{loc}$ coincides with the Orlicz space $L^1 \log L^1$ . Using the Fefferman-Stein characterization of Hardy, under the only assumption that $\nabla a$ and $\nabla b$ are in $L^2$, one deduces that the convolution between the Green Kernel $\log |x|$ and the jacobian (1.1) is in the Sobolev space $W^{2,1}$.

These improvements in integrability or regularity were originally obtained together with estimates. We sumarize then the previous discussion and give the corresponding estimates in the following theorem

***Theorem 1.1* (Wente 1969, Tartar 1983, Müller 1989, Coiffman-Lions-Meyer-Semmes 1990)**

*Let $\omega$ be a bounded regular domain of $\mathbb{R}^2$. Let $a$ and $b$ be two measurable functions on $\omega$ whose gradients are in $L^2(\omega)$. Then there exists a unique solution $\varphi$ in $W^{1,2}(\omega)$ to*

$$\left\{\begin{array}{ll} -\Delta\varphi = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x} & \text{in } \omega \\ \varphi = 0 & \text{on } \partial\omega \end{array}\right. \tag{1.2}$$

*Moreover there exists a constant $C(\omega)$ such that*

$$\|\varphi\|_{L^\infty(\omega)} + \|\nabla\varphi\|_{L^{2,1}(\omega)} + \|\nabla^2\varphi\|_{L^1(\omega)} \le C(\omega)\ \|\nabla a\|_{L^2}\ \|\nabla b\|_{L^2}. \tag{1.3}$$

*In particular $\varphi$ is continuous in $\omega$.*

A consequence of the previous theorem was obtained by Fabrice Bethuel in [2] using a duality argument and is another remarquable result in the theory of integrablity by compensation (see also a presentation of this result in [19]) .

***Theorem 1.2* (Bethuel 1992).** — *Let $\omega$ be a bounded regular domain of $\mathbb{R}^2$. Let $a$ and $b$ be two measurable functions on $\omega$. Assume that the distributional derivatives $\nabla a$ and $\nabla b$ are respectively in $L^2(\omega)$ and $L^{2,\infty}(\omega)$. Then there exists a unique solution $\varphi$ in $W^{1,2}(\omega)$ to*

$$\left\{\begin{array}{ll} -\Delta\varphi = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x} & \text{in } \omega \\ \varphi = 0 & \text{on } \partial\omega \end{array}\right. \tag{1.4}$$

*Moreover there exists a constant $C(\omega)$ such that*

$$\|\nabla\varphi\|_{L^2(\omega)} \le C(\omega)\ \|\nabla a\|_{L^2(\omega)}\ \|\nabla b\|_{L^{2,\infty}(\omega)}. \tag{1.5}$$

## 2. Hildebrandt's conjecture on critical points to conformally invariant Lagrangians

**2.1. Conformally invariant quadratic coercive lagrangians in 2 dimensions.** — Due to the important role they play in physics and geometry, the analysis of critical points to conformally invariant lagrangians has raised a special interest in the mathematical community since at least the early 50's and in particular under the impulsion of Charles B. Morrey. Because of the richness of it's conformal group, the dimension 2 should maybe be first looked at in priority.

Let first consider the Dirichlet Energy for functions $u$ from a 2-dimensional domain $\omega$ into $\mathbb{R}$

$$L(u) := \int_\omega |\nabla u|^2(x,y)\ dx \wedge dy.$$

This is maybe the most simple example of a 2-dimensional conformally invariant Lagrangian. Indeed, let $\phi$ be a conformal transformation on $\mathbb{C}$, it satisfies

$$\left\{\begin{array}{l} \left|\dfrac{\partial\phi}{\partial x}\right| = \left|\dfrac{\partial\phi}{\partial y}\right|, \\[2ex] \dfrac{\partial\phi}{\partial x}\cdot\dfrac{\partial\phi}{\partial y} = 0, \\[2ex] det\,\nabla\phi \geq 0 \qquad \text{and} \qquad \nabla\phi \neq 0. \end{array}\right. \tag{2.1}$$

(In other words $\phi$ is an holomorphic function). Then, for any $u$ in $W^{1,2}(\omega,\mathbb{R})$ the following holds

$$L(u) = L(u\circ\phi) = \int_{\phi^{-1}(\omega)} |\nabla(u\circ\phi)|^2(x,y)\ dx\wedge dy. \tag{2.2}$$

Critical points to $L$ for any perturbation of the form $u + t\chi$, where $\chi$ is an arbitrary smooth compactly supported function on $\omega$, are the harmonic functions satisfying

$$\Delta u = 0 \quad \text{dans } \omega. \tag{2.3}$$

Among the analysis questions related to that functional come first the regularity issues, uniqueness questions for fixed given boundary data, questions regarding the shape of the solution—possible symmetries, etc. In that elementary situation one can observe that most of these priority questions regarding the analysis of solution to (2.3) can be solved by the mean of the maximum principle.

The previous problem can be generalized in the following ways. We can first extend $L$ to maps taking values into an arbitrary Euclidian space $\mathbb{R}^n$ as follows

$$L(u) := \int_\omega |\nabla u|^2(x,y)\ dx\wedge dy = \int_\omega \sum_i |\nabla u^i|^2\ dx\wedge dy.$$