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AN INVITATION TO p-ADIC DIFFERENTIAL EQUATIONS

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AN INVITATION TO p-ADIC DIFFERENTIAL EQUATIONS

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Abstract. — A reworking of an introductory course on p-adic linear differential equations.

***Résumé* (Une invitation aux équations différentielles p-adiques).** — Une rielaboration d'une cours propedeutique sur les équations différentielles lineares p-adiques.

1. Introduction

These notes are a more extensive version[(1)] of an introductory course on p-adic differential equations given by the author[(2)] at Luminy in September 2009. The aim is to acquaint those versed in classical differential equations with the main new flavors to be tasted in the p-adic setting. We have sought not only to draw the parallels between well known features of the classical apparatus of differential equations (such as the Cauchy theorem, the notion of irregularity, and the definition of regular singularities) and their p-adic analogues, but also to elucidate non-classical phenomena which arise only in the p-adic setting. The latter exhibit important differences with respect to classical behavior. It is hoped that this method will aid the reader in understanding important recent achievements in p-adic differential equations obtained by Christol, Kedlaya and Mebkhout (among others) in following the path initiated by Dwork and Robba. We have deliberately left aside one specific tool which is a key ingredient for the study of p-adic differential equations: the Frobenius structure. Although the Frobenius structure is indeed a basic tool in the the theory of p-adic differential

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equations and appears in almost all the deeper proofs (as we tried to underline in the present paper), it is also far removed from the classical case. In our opinion, dedicating the necessary space to its presentation at this initial level would detract the reader's attention from the central achievements expounded here. Moreover, in the near future at least two more detailed works will be published on the subject of p-adic differential equations ([33] and a forthcoming book by Christol [18]). They will cover all the advanced results on the subject. Thus these notes should be understood as an invitation and aid to the non-specialist desiring to approach the subject matter rather than a comprehensive introduction to p-adic differential equations.

We give a brief summary of the sequel. In §2 we introduce the main objects of study, ultrametric fields. In §3 we recall a few standard facts on differential operators, and give some examples to illustrate differences between the classical and ultrametric settings. The p-adic exponential function and some related power series are introduced in §4. The section concludes with an application of the Dieudonné–Dwork theorem to the Artin-Hasse exponential function. In §5 we give some very brief remarks regarding analyticity for p-adic functions, and introduce some rings of functions used in the sequel. §6 reviews the classical iterative procedure for obtaining a formal matrix solution of a linear differential system, thus raising the problem of finding the radius of convergence for such solutions. The section concludes with the ultrametric analogue of Cauchy's estimate for such radii of convergence. Again in §7 we review some classical notions (logarithmic derivatives, indicial polynomial and exponents at a singular point) and obtain a p-adic analogue of Cauchy's estimate for the radius of convergence of formal solutions. In §8 we review the classical definition of irregularity and Malgrange's theorem, and offer an example to show the relevance of "arithmetic" properties of the coefficients of a p-adic differential operator. To that end we introduce the notion of the type of a number, and discuss the relevance of p-adic Liouville numbers in problems of convergence of solutions. §9 provides a categorical setting for the study of differential modules M over a differential ring R, and gives matricial interpretations for various concepts when the module M is a finitely generated free R-module. §10 starts with some remarks on the boundary behavior of p-adic Laurent series, in particular to p-adic analogues of Cauchy's estimates and the Maximum Modulus theorem. These considerations lead naturally to the introduction of the notions of ultrametric norms and seminorms. The section concludes with some examples which are used later. §11 introduces the notion of multiplicative norms as "points", a key notion for Berkovich's work on analytic spaces. §12 discusses the radius of convergence a p-adic linear system of differential equations, and its relation to the spectral norm of differential operators. §13 gives a discussion of transfer theorems, the definition of logarithmic growth for solutions of differential systems, and the Dwork-Robba theorem. §14 discusses the variation of the radius of convergence of solutions of p-adic linear systems of differential equations, optimal bases, Young's theorem, and the Amice-Fontaine ring $\mathcal{E}$, and a result of Christol-Mebkhout on the last slope of the Newton polygon of the system. The main result discussed in §15 is a theorem of Dwork and Robba on the decomposition of a differential module according to the radii of convergence of its solutions at the generic point. In §16 we deal with the problem of decomposing solvable differential modules, that is, modules all of whose solutions have the same radius of convergence at the generic point. We also review the Christol-Mebhkout approach for decomposing non-solvable modules, and conclude with a few remarks on the ultrametric analogue of the classical notion of irregularity. §17 gives some suggestions for further reading on related topics. Finally, we would like to thank Bernard Le Stum.

2. Ultrametric fields

Our objects of study will be differential equations over ultrametric fields and their solutions. In this section we review the standard concepts and definitions. For further details see [28].

An *ultrametric field* K is a field endowed with an *ultrametric absolute value*, that is, a function

$$|\cdot| : K \longrightarrow \mathbb{R}^{\geq 0}$$

satisfying

1. $|a| = 0$ if and only if $a = 0$
2. $|ab| = |a|\,|b|$
3. $|a+b| \leq \max\{|a|, |b|\}$

for all $a, b \in K$.

Note that 3. is a stronger condition than the usual triangle inequality.

A logarithmic version of the absolute value is called an *order function*

$$\mathrm{ord} : K \longrightarrow \mathbb{R} \cup \{\infty\}.$$

For any fixed real number c with $0 < c < 1$ one can pass between an order function and the corresponding absolute value on the field via the relation

$$|a| = c^{\mathrm{ord}\, a}$$

The ring

$$\mathcal{V}_K := \left\{x \in K \,\middle|\, |x| \leq 1\right\}$$

is called the *ring of integers* of K.

The set

$$\mathfrak{p}_K := \left\{x \in K \,\middle|\, |x| < 1\right\}$$

is a maximal ideal of $\mathcal{V}_K$. The field $\mathcal{V}_K/\mathfrak{p}_K$ is called the *residue field* of K.

Example 2.1. — Let p be a fixed rational prime and let $a \in \mathbb{Q}$. Write $a = p^{\ell} m/n$ with $(m, p) = (n, p) = 1$. Then define

$$\mathrm{ord}_p(a) = \ell.$$

Example 2.2. — Let $k(x)$ be the field of rational functions with coefficients in the field k, and let $\phi(x) \in k(x)$. Write $\phi(x) = x^\ell f(x)/g(x)$ with $f(0)g(0) \neq 0$. Then we define

$$\operatorname{ord}_x(\phi(x)) = \ell.$$

In both examples it is easy to verify the properties of the order function. In the first example, the usual constant chosen to pass to the corresponding absolute value is $c = 1/p$. Thus,

$$|a|_p = p^{-\operatorname{ord}_p(a)}.$$

Usually we suppress the subscript p on the absolute when there is no danger of confusion with the archimedean absolute value.

If K is an ultrametric field, the absolute value defines a metric topology on K. If $x_0 \in K$, and $r \in \mathbb{R}^{\geq 0}$ then one defines

$$D_K(x_0, r^-) = D(x_0, r^-) = \{x \in K |\, |x - x_0| < r\}$$

and

$$D_K(x_0, r^+) = D(x_0, r^+) = \{x \in K |\, |x - x_0| \leq r\}.$$

One must, however, be aware that for ultrametric fields "familiar concepts" assume unfamiliar properties. For example, for such K the "open ball" $D(x_0, r^-)$ is both open and closed in the valuation topology, while the "closed ball" is both closed and open, as also is the "circle" $\{x |\, |x| = r\}$. Indeed, when $r = 1$ the "circle" $\{x |\, |x| = 1\}$ consists of the union of the open balls $D(x, 1^-)$ where x runs over a system of representatives for the residue field of $K\backslash\{0\}$. Thus K is totally disconnected in the topology induced by $|\cdot|$. Furthermore, one shows easily that if $D(x_0, r)$ and $D(x_1, r')$ are two ultrametric disks, then $D(x_0, r) \cap D(x_1, r') = \varnothing$ except when $D(x_0, r) \subset D(x_1, r')$ or $D(x_1, r') \subset D(x_0, r)$, a fact which precludes the classical approach to analytic continuation. Moreover, the following facts are easy consequences of the strong triangle inequality:

1. All "triangles" in K are isosceles with "third" side of "length" less than or equal to that of the equal sides.
2. Every point in a disk $D(x_0, r^-)$ is a center for the disk, in the sense that if $x_1 \in D(x_0, r^-)$ then $D(x_1, r^-) = D(x_0, r^-)$, and similarly for "closed disks" $D(x_0, r^+)$.
3. $D(0, 1^+) \supset \mathbb{Z}1_K$ (which coincides with $\mathbb{Z}$ for K of characteristic zero).

An ultrametric field K is said to have a *discrete valuation*, if the group of values $|K^\times|$ (where $K^\times = K\backslash\{0\}$) is a discrete subgroup of the $\mathbb{R}^{>0}$. An ultrametric field is said to be *complete* if it is complete as a metric space, that is, every Cauchy sequence in K converges to an element of K. Note that the strong triangle inequality implies that a sequence $\{a_n\}$ in K is a Cauchy sequence if and only if $|a_n - a_{n+1}| \to 0$ for $n \to \infty$. As a consequence, over a complete ultrametric field a series

$$\sum_{i=1}^{\infty} a_i$$