Séminaires & Congrès

COLLECTION SMF

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ARITHMETIC AND GALOIS THEORY OF DIFFERENTIAL EQUATIONS

Numéro 23

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
23, 2011, p. 169–227

DWORK'S 1994 PADOVA LECTURES ON EXPONENTIAL MODULES

by

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Abstract. — A reworking of Dwork's 1994 Padova lectures on exponential modules.

***Résumé* (Leçons de Dwork sur le modules exponentielles).** — Une relecture des notes d'un cours donné à Padoue en 1994.

1. Introduction

The present text is a reworking of an introductory course on exponential modules given at Padova during the spring of 1994. The aim of the course was to prove in full detail the fact announced in the Introduction of [11] that exponential modules are examples of G–modules. These lectures have been the starting point for a number of later generalizations: [5] and [6]. The present article should prove of interest in clarifying the general case and motivating the basic underlying constructions. In fact, herein one finds all the "classical" Dwork techniques: Frobenius, deformations and the use of generic results to study the effect of specialization. However, here we see also the starting point for a generalization of the π of Dwork (section 10) which will be developed by [7] and [14]. Our work started from some notes, taken by Sullivan and Gerotto, from Dwork's 1994 course. We used them as a skeleton for our text. We had to fill several gaps and to add and cut several parts: it was Dwork's particular habit to pursue an idea one week with the aim of proving some desired result, and then to cast it aside or substantially modify it the next week in favor of a more effective approach. In editing these notes we have tried to retain the sometimes rather colorful style of Dwork's original lectures. But of course we are unable to match his mathematical skill: all errors or obscurities in what follows should be attributed to the redactors.

2000 Mathematics Subject Classification. — 11D88, 34M03,34M15,34M35.
Key words and phrases. — Exponential Module, G-function.

B. Chiarellotto was supported by a Cariparo Eccellenza Grant "Differential methods on algebra, geometry and arithmetic".

Notations and terminology not explicitly defined here are those used in [11], which will be cited frequently in the sequel.

Here is an outline of the article: in Section 2 and 3 definitions and very elementary examples of exponential modules are given, and the relation between exponential modules and classical cohomology spaces via the algebraic Laplace transform is explained in the simplest case. Section 4 deals with Koszul techniques which are used to prove finiteness of exponential modules in the regular case. The "resultant" which will play a fundamental role in the sequel is also introduced there. Section 5 introduces the dual theory and gives an explicit description of the differential equation related to an exponential module. Section 6 presents the statement of the main result and a sketch of the lines of the proof which will be carried out in Sections 7, 8, 9, 10. Section 11 treats an estimate of the global inverse radius in the regular case.

2. Exponential Modules

Let K be an algebraic number field, $[K:\mathbb{Q}] < \infty$, and let λ be transcendental over K. Put $R = K(\lambda)[X_1, \ldots, X_{n+1}]$ and for $i = 1, 2, \ldots, n+1$ let E_i be the differential operator defined by

$$E_i = X_i \frac{\partial}{\partial X_i}. \tag{1}$$

For $f(X_1, \ldots, X_{n+1}) \in R$ we define $f_i \in R$ by

$$f_i = E_i f, \tag{2}$$

and a new differential operator D_i by

$$D_i = E_i + f_i. \tag{3}$$

Note that R is stable under each D_i and that

$$D_i \circ D_j = D_j \circ D_i \tag{4}$$

for $i, j = 1, 2, \ldots, n+1$. Moreover, as one easily verifies, each of the operators D_i annihilates $\exp(-f(X_1, \ldots, X_{n+1})) \in K(\lambda)[[X_1, \ldots, X_{n+1}]]$.

Definition 2.1. — *An* exponential module *is a $K(\lambda)$–module of the form*

$$\mathcal{W} = R \Big/ \sum_{i=1}^{n+1} D_i R.$$

We further note that if we set

$$\sigma_\lambda = \frac{\partial}{\partial \lambda} + \frac{\partial f}{\partial \lambda}, \tag{5}$$

then R is stable under σ_λ and $\sigma_\lambda(\exp -f(X_1, \ldots, X_{n+1})) = 0$. From a formal point of view, each of the operators D_i and σ_λ may be viewed as a "twisted" form of partial differentiation:

$$D_i = \frac{1}{\exp f} \circ E_i \circ \exp f$$

$$\sigma_\lambda = \frac{1}{\exp f} \circ \frac{\partial}{\partial \lambda} \circ \exp f$$

which both gives a rapid proof of (4) and shows that

$$\sigma_\lambda \circ D_i = D_i \circ \sigma_\lambda \tag{6}$$

for $i = 1, \ldots, n+1$. From this and the stability of R under σ_λ it follows that σ_λ induces a *connection* on $\mathcal{W}$ (again denoted by σ_λ) which extends the action of $d/d\lambda$ on $K(\lambda)$, that is a K–linear map

$$\sigma_\lambda \colon \mathcal{W} \longrightarrow \mathcal{W}$$

such that for $h \in K(\lambda)$ and $w \in \mathcal{W}$ we have

$$\sigma_\lambda(hw) = \frac{dh}{d\lambda} w + h\sigma_\lambda(w).$$

It is also true, as we shall see, that

$$\dim_{K(\lambda)} \mathcal{W} < \infty,$$

so that $\mathcal{W}$ is a differential module over $K(\lambda)$ with respect to the action of σ_λ.

Definition 2.2. — *The differential module $\mathcal{W}$ is called a G–*module *if the solutions of the differential system corresponding to the action of σ_λ on $\mathcal{W}$ are G–functions.*

In general, it is not true that every exponential module is a G–module. If, however, the polynomial f used in the definition of $\mathcal{W}$ is a homogeneous polynomial in $X_1, \ldots, X_{n+1}$ then we will show that $\mathcal{W}$ is a *G–module*. Indeed it is possible to weaken the homogeneity condition in various ways. For example, if f is of the form $X_{n+1}\, g(X_1, \ldots, X_n)$ then $\mathcal{W}$ is always a G–module. More generally (and with a slight change in the indexing) if $f \in K(\lambda)[X_1, \ldots, X_k, X_{k+1}, \ldots, X_{k+l}]$ has the form

$$f = X_{k+1} f^{(1)}(X_1, \ldots, X_k) + \cdots + X_{k+l} f^{(l)}(X_1, \ldots, X_k)$$

with $f^{(j)}(X_1, \ldots, X_k) \in K(\lambda)[X_1, \ldots, X_k]$, for $j = 1, \ldots, l$, and if the operators D_i are defined by

$$D_i = X_i \frac{\partial}{\partial X_i} + X_i \frac{\partial f}{\partial X_i}$$

then the corresponding $\mathcal{W}$ is again a G–module. With some additional technical conditions one can also handle the case in which $\mathcal{W}$ is defined by operators D_i of the type

$$D_i = E_i + f_i + a_i$$

with $a_i \in \mathbb{Q}$. We now illustrate the foregoing discussion by means of some very simple examples.

Example 1. — We consider first the case in which $R = K(\lambda)[X]$ and $f = X^2+2\lambda X+1$. Here we need only deal with $E = E_1 = X\frac{\partial}{\partial X}$. We also set $D = D_1$. We have

$$f_1 = E_1 f = 2X^2 + 2\lambda X.$$

For any integer $m \geq 2$ we have the following reduction law

$$(7) \qquad X^m = X^{m-2}\frac{1}{2}f_1 - X^{m-2}(\lambda X) = D(\frac{1}{2}X^{m-2}) - E(\frac{1}{2}X^{m-2}) - \lambda X^{m-1}$$
$$= -\frac{1}{2}(m-2)X^{m-2} - \lambda X^{m-1} + D(\frac{1}{2}X^{m-2}).$$

It then follows that $1, X$ represent a basis of $\mathcal{W}$, that is

$$\mathcal{W} = R/DR \cong K(\lambda)\,1 + K(\lambda)\,X.$$

Moreover we have

$$\sigma_\lambda = \partial/\partial\lambda + 2X$$

and

$$\sigma_\lambda \begin{pmatrix} 1 \\ X \end{pmatrix} = \begin{pmatrix} 2X \\ 2X^2 \end{pmatrix} = \begin{pmatrix} 2X \\ -2\lambda X \end{pmatrix} \quad (\text{mod } DR),$$

since for $m = 2$ equation (7) gives $X^2 = -\lambda X$ (mod DR). Therefore the action of σ_λ is represented by

$$\sigma_\lambda \begin{pmatrix} 1 \\ X \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & -2\lambda \end{pmatrix} \begin{pmatrix} 1 \\ X \end{pmatrix}.$$

Thus if we consider the associated differential system

$$\frac{d}{d\lambda}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & -2\lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

we find that

$$\frac{y_2'}{y_2} = -2\lambda$$

so that $\log y_2 = -\lambda^2 + C$, that is

$$y_2 = \frac{C_1}{\exp(\lambda^2)}.$$

Clearly the singularity of this system at $\lambda = \infty$ is not a regular singularity, so the solution element y_2 is not a G–function in λ.

This example also illustrates a general phenomenon: a (reducible) system corresponding to a matrix of the form

$$G = \begin{pmatrix} G_1 & G_2 \\ 0 & G_4 \end{pmatrix}$$

where $G_1 \in \mathcal{M}_{m_1}(K(\lambda))$ and $G_4 \in \mathcal{M}_{m_2}(K(\lambda))$ will define a G–module if both G_1 and G_4 correspond to systems defining G–modules.