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NOTES ON A-HYPERGEOMETRIC FUNCTIONS

by

F. Beukers

Abstract. — These notes are a first introduction to the theory of A-hypergeometric functions introduced by Gel'fand, Kapranov and Zelevinsky at the end of the 1980's. These functions subsume earlier classes of hypergeometric functions in one and in several variables, like the hypergeometric functions named after Gauss, Appell, Horn, Lauricella, Aomoto-Gel'fand.

***Résumé* (Notes sur les fonctions A-hypergéométriques).** — Ces notes constituent une première introduction à la théorie des fonctions A- hypergéométriques, introduites par Gel'fand, Kapranov et Zelevinsky à la fin des années 80. Ces fonctions englobent des nombreuses classes de fonctions hypergéométriques de une ou plusieurs variables définies antérieurement, telles que les fonctions hypergéométriques de Gauss, Appell, Horn, Lauricella et Aomoto-Gel'fand respectivement.

1. Introduction

Hypergeometric functions of Gauss type are immediate generalisations of the classical elementary functions like sin, arcsin, arctan, log, etc. They were studied extensively in the 19th century by mathematicians like Kummer and Riemann. Towards the end of the 19th century and the beginning of the 20th century hypergeometric functions in several variables were introduced. For example Appell's functions, the Lauricella functions and the Horn series. Around 1990, in the series of papers [16], [21], [19], [17] it was realised by Gel'fand, Kapranov and Zelevinsky that all above types and their differential equations fit into a far more general but extremely elegant scheme of so-called A-hypergeometric functions, or GKZ-hypergeometric functions.

Nowadays hypergeometric functions of all types (including GKZ-type, but also many others not mentioned here) are ubiquitous throughout the mathematics and

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mathematical physics literature, ranging from orthogonal polynomials, modular forms to scattering theory and mirror symmetry.

The present notes form an introduction to A-hypergeometric functions. We describe their defining equations and explicit solutions in the form of power series expansions and so-called Euler integral representations. We also discuss the associated D-modules and their relation with the work of B.Dwork in [12]. The latter book describes a theory of generalised hypergeometric functions which runs for a large part in parallel with the theory of Gel'fand, Kapranov and Zelevinsky. However, the language is entirely different and a large part of [12] is also devoted to the p-adic theory of generalised hypergeometric functions.

Essentially the first book devoted entirely to A-hypergeometric functions is the one by Saito, Sturmfels and Takayama [31]. In addition, there are several introductory notes such as [34] and [30], discussing similar, and on the other hand, different aspects of the theory. The book [36] deals with a certain type of A-hypergeometric function, namely the Aomoto-systems $E(2,4)$ and $E(3,6)$. However, it does cover aspects such as monodromy calculations for this system and a moduli interpretation of the underlying geometry. These subjects are not addressed in this survey, simply because a general theory is still lacking. In a forthcoming publication we like to show how subgroups of the monodromy group of general A-hypergeometric systems can be computed.

Another aspect not dealt with in these notes is the question which hypergeometric equations have all of their solutions algebraic over the rational function field generated by their variables. This is a classical question. In 1873 H.A.Schwarz compiled his famous list of Gauss hypergeometric functions which are algebraic. This list was extended to general one variable hypergeometric functions in 1989 by G.Heckman and F.Beukers in [7]. In the several variable case Schwarz's list had also been extended to functions such as Appell's F1 (T.Sasaki, [32]), Appell F2 (M.Kato, [24]), Appell F4 (M.Kato, [23]), Lauricella's FD (Cohen-Wolfart, [3]) and the Aomoto system $X(3,6)$ (K.Matsumoto, T.Sasaki, N.Takayama, M.Yoshida, [27]).

In 2006 the present author found a combinatorial characterisation for algebraic A-hypergeometric functions (in the irreducible case) in [5]. It is perhaps interesting to note that, as an application, Esther Bod [9] succeeded in extending Schwarz's list to all irreducible Appell, Lauricella and Horn equations.

Finally, we should mention the book of Gel'fand, Kapranov and Zelevinsky, [18], which is not on A-hypergeometric functions proper, but on A-resultants and discriminants which arise in connection the singular loci of A-hypergeometric systems.

2. The one variable case

Let $\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n$ be any complex numbers and consider the generalised hypergeometric equation in one variable,

$$z(D+\alpha_1)\cdots(D+\alpha_n)F=(D+\beta_1-1)\cdots(D+\beta_n-1)F,\qquad D=z\frac{d}{dz} \tag{1}$$

This is a Fuchsian equation of order n with singularities at $0, 1, \infty$. The local exponents read,

$$\begin{array}{ll} 1-\beta_1, \ldots, 1-\beta_n & \text{at } z=0 \\ \alpha_1, \ldots, \alpha_n & \text{at } z=\infty \\ 0, 1, \ldots, n-2,\ -1+\sum_1^n(\beta_i-\alpha_i) & \text{at } z=1 \end{array}$$

When the β_i are distinct modulo 1 a basis of solutions at $z = 0$ is given by the functions

$$z^{1-\beta_i}\ {}_nF_{n-1}\left(\begin{array}{c} \alpha_1-\beta_i+1, \ldots, \alpha_n-\beta_i+1 \\ \beta_1-\beta_i+1, ..^{\vee}.., \beta_n-\beta_i+1 \end{array}\middle| z\right) \qquad (i=1,\ldots,n).$$

Here $..^{\vee}..$ denotes suppression of the term $\beta_i - \beta_i + 1$ and ${}_nF_{n-1}$ stands for the generalised hypergeometric function in one variable

$${}_nF_{n-1}\left(\begin{array}{c} \alpha_1, \ldots, \alpha_n \\ \beta_1, \ldots, \beta_{n-1} \end{array}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_{n-1})_k k!} z^k.$$

Here $(x)_k$ is the Pochhammer symbol defined by $(x)_k = \Gamma(x+k)/\Gamma(x) = x(x+1)(x+2)\cdots(x+k-1)$. The function $\Gamma(z)$ is of course the Euler Γ-function. When the α_j are distinct modulo 1 we have the following n independent power series solutions in $1/z$,

$$z^{-\alpha_j}\ {}_nF_{n-1}\left(\begin{array}{c} \alpha_j-\beta_1+1, \ldots, \alpha_j-\beta_n+1 \\ \alpha_j-\alpha_1+1, ..^{\vee}.., \alpha_j-\alpha_n+1 \end{array}\middle| \frac{1}{z}\right) \qquad (j=1,\ldots,n).$$

At $z = 1$ we have the following interesting situation.

***Theorem 2.1* (Pochhammer).** — *The equation (1) has $n - 1$ independent holomorphic solutions near $z = 1$.*

However, the solutions are not as easy to write down.

Finally we mention the Euler integral for ${}_nF_{n-1}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_{n-1}|z)$,

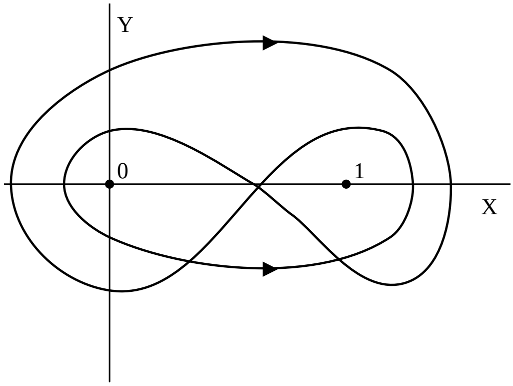
$$\prod_{i=1}^{n-1} \frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)} \int_0^1 \cdots \int_0^1 \frac{\prod_{i=1}^{n-1} t_i^{\alpha_i-1}(1-t_i)^{\beta_i-\alpha_i-1}}{(1-zt_1\cdots t_{n-1})^{\alpha_n}} dt_1 \cdots dt_{n-1}$$

for all $\Re\beta_i > \Re\alpha_i > 0$ $(i = 1, \ldots, n-1)$.

In the case $n = 2$ this becomes the famous Euler integral

$${}_2F_1(a, b, c|z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt \quad (\Re c > \Re b > 0)$$

The restriction $\Re c > \Re b > 0$ is included to ensure convergence of the integral at 0 and 1. We can drop this condition if we take the Pochhammer contour γ given by



as integration path. Notice that the integrand acquires the same value after analytic continuation along γ.

It is a straightforward exercise to show that for any $b, c-b \notin \mathbb{Z}$ we have

$$ {}_2F_1(a,b,c|z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{1}{(1-e^{2\pi i b})(1-e^{2\pi i(c-b)})} \int_\gamma t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt. $$

In Section 20 we shall generalise the Pochhammer contour to higher dimensional versions.

3. Appell and Lauricella functions

There exist many generalisations of hypergeometric functions in several variables. The most well-known ones are the Appell functions in 2 variables, introduced by P.Appell in 1880, and Lauricella functions of n variables. The Appell functions read

$$ \begin{aligned} F_1(a,b,b',c,x,y) &= \sum \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!} x^m y^n \\ F_2(a,b,b',c,c',x,y) &= \sum \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n m!n!} x^m y^n \\ F_3(a,a',b,b',c,x,y) &= \sum \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}m!n!} x^m y^n \\ F_4(a,b,c,c',x,y) &= \sum \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n m!n!} x^m y^n \end{aligned} $$

The guiding principle for these functions is the following. Consider the product

$$ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix} \middle| x\right) {}_2F_1\left(\begin{matrix} a',b' \\ c' \end{matrix} \middle| y\right) = \sum \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_m(c')_n m!n!} x^m y^n. $$

Now replace one or two of the product pairs

$$ (a)_m, (a')_n \quad (b)_m(b')_n \quad (c)_m(c')_n $$

by the corresponding

$$ (a)_{m+n}, \quad (b)_{m+n}, \quad (c)_{m+n}. $$