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MULTIVARIATE p-ADIC FORMAL CONGRUENCES AND INTEGRALITY OF TAYLOR COEFFICIENTS OF MIRROR MAPS

by

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Abstract. — We generalise Dwork's theory of p-adic formal congruences from the univariate to a multivariate setting. We apply our results to prove integrality assertions on the Taylor coefficients of (multivariable) mirror maps. More precisely, with $\mathbf{z} = (z_1, z_2, \ldots, z_d)$, we show that the Taylor coefficients of the multivariable series $q(\mathbf{z}) = z_i \exp(G(\mathbf{z})/F(\mathbf{z}))$ are integers, where $F(\mathbf{z})$ and $G(\mathbf{z}) + \log(z_i)F(\mathbf{z})$, $i = 1, 2, \ldots, d$, are specific solutions of certain GKZ systems. This result implies the integrality of the Taylor coefficients of numerous families of multivariable mirror maps of Calabi–Yau complete intersections in weighted projective spaces, as well as of many one-variable mirror maps in the "*Tables of Calabi–Yau equations*" [`ar`χ`iv:math/0507430`] of Almkvist, van Enckevort, van Straten and Zudilin. In particular, our results prove a conjecture of Batyrev and van Straten in [*Comm. Math. Phys.* **168** (1995), 493–533] on the integrality of the Taylor coefficients of canonical coordinates for a large family of such coordinates in several variables.

***Résumé* (Congruences multivariÃľes p-adiques formelles et intÃľgralitÃľ des coefficients de Taylor des applications miroir)**

Nous généralisons en plusieurs variables la théorie de Dwork sur les congruences formelles p-adiques en une variable. Nous appliquons nos résultats à la preuve de l'intégralité des coefficients de Taylor d'applications miroir de plusieurs variables. Plus précisément, en notant $\mathbf{z} = (z_1, z_2, \ldots, z_d)$, nous montrons que les coefficients de Taylor des séries de plusieurs variables $q(\mathbf{z}) = z_i \exp(G(\mathbf{z})/F(\mathbf{z}))$ sont des entiers, où $F(\mathbf{z})$ et $G(\mathbf{z}) + \log(z_i)F(\mathbf{z})$, $i = 1, 2, \ldots, d$, sont des solutions spécifiques de certains systèmes GKZ. Ce résultat implique l'intégralité des coefficients de Taylor de nombreuses familles d'applications miroir (de plusieurs variables) d'intersections complètes de type Calabi–Yau dans des espaces projectifs à poids, ainsi que ceux de

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nombreuses applications miroir d'une variable dans la table "*Tables of Calabi–Yau equations*" [arχiv:math/0507430] de Almkvist, van Enckevort, van Straten et Zudilin. En particulier, nos résultats démontrent une conjecture de Batyrev et van Straten [*Comm. Math. Phys.* **168** (1995), 493–533] concernant l'intégralité des coefficients de Taylor des coordonnées canoniques pour une large classe de telles coordonnées en plusieurs variables.

1. Introduction and statement of the results

In [7, 8, 9, 10, 11], Dwork developed a sophisticated theory for proving analytic and arithmetic properties of solutions to (p-adic) differential equations. In [7, 10], he focussed on the case of hypergeometric differential equations. In particular, the article [10] contains a "formal congruence" criterion that enabled him to address the analytic continuation of quotients of certain solutions and to establish arithmetic properties satisfied by exponentials of such quotients. These exponentials of ratios of solutions to hypergeometric differential equations (in fact, of Picard–Fuchs equations) have recently received great attention in mathematical physics and algebraic geometry under the name of *canonical coordinates*. Their compositional inverses, known as *mirror maps*, are an important ingredient in the computation of the Yukawa coupling in the theory of mirror symmetry. It is conjectured that the coefficients in the Lambert series expansion of the Yukawa coupling produce Gromov–Witten invariants of classes of rational curves.

It is only relatively recently that Dwork's theory has been systematically applied to obtain general arithmetic results on the Taylor coefficients of mirror maps. Partial results in this direction were found by Lian and Yau [18, 19], by Zudilin [22], and by Kontsevich, Schwarz and Vologodsky [13, 21]. The (so far) strongest and most general results are contained in [6, 14, 15], where, in particular, numerous integrality results for the Taylor coefficients of univariate mirror maps of Calabi–Yau complete intersections in weighted projective spaces are proven, improving and refining the afore-mentioned results by Lian and Yau, and by Zudilin. However, all these results do not touch the case of *multivariable* mirror maps, upon which they are not able to say anything. The goal of this paper is to set the basis of a theory which is capable to address questions of integrality of Taylor coefficients of multivariable mirror maps, and to apply this theory systematically to large classes of such mirror maps.

1.1. Multivariate theory of formal congruences. — The proof strategy in [6, 14, 15, 18, 19, 22] for obtaining integrality assertions on the Taylor coefficients of one-variable mirror maps is crucially based on a series of reductions and results, of which the corner stones are:

(D1) the conversion of the integrality problem to a p-adic problem;

(D2) a lemma due to Dieudonné and Dwork (cf. [17, Ch. 14, p. 76]) providing a criterion for deciding whether a power series with coefficients over $\mathbb{Q}_p$ has coefficients in $\mathbb{Z}_p$;
(D3) a reduction lemma for harmonic numbers due to the authors (cf. [15, Lemma 1, respectively Lemma 5] and [14, Lemma 3]);
(D4) a combinatorial lemma due to Dwork [10, Lemma 4.2] for rearranging sums that appear in this context in a way tailor-made for p-adic analysis;
(D5) Dwork's theorem on formal congruences (cf. [10, Theorem 1.1]).

We point out that Lian and Yau, and Zudilin do not need item (D3) due to the nature of the special families of mirror maps that they were considering. Indeed, item (D3) is the decisive novelty which enabled the authors to arrive at their general sets of results in [14, 15]. We remark that Zudilin also condenses (D4) and (D5) into one step in the proof of his main result in [22]. However, in order to arrive at the general results in [14, 15], it turned out to be necessary to follow the full path outlined by (D1)–(D5) above, as attempts to lift Zudilin's variation to this generality failed.

With the exception of (D1), which trivially extends to the multivariable case, for none of the above items there exist multivariate extensions in the current literature. In particular, no approach for attacking integrality questions for multivariable mirror maps has been available so far.

In this paper, we present multivariate versions for all of (D2)–(D5); all of them seem to be new. Our multivariate extension of (D2) is the content of Lemma 2.1 in Section 2, our multivariate version of (D3) can be found in Lemma 2.3 in Section 2, while Lemma 6.1 in Section 6 provides our multivariate extension of (D4). On the other hand, we state our multivariate extension of item (D5) in Theorem 1.1 below. Since its one-variable special case enabled Dwork to address the question of analytic extension of certain ratios of generalised p-adic hypergeometric series in one variable, we expect our result below to be the appropriate tool for analogous studies of multivariable p-adic hypergeometric series.

For the statement of our multivariate theorem on formal congruences, we need some standard multi-index notation. Namely, given a positive integer d, a real number λ, and vectors $\mathbf{m} = (m_1, m_2, \dots, m_d)$ and $\mathbf{n} = (n_1, n_2, \dots, n_d)$ in $\mathbb{R}^d$, we write $\mathbf{m} + \mathbf{n}$ for $(m_1 + n_1, m_2 + n_2, \dots, m_d + n_d)$, $\lambda\mathbf{m}$ for $(\lambda m_1, \lambda m_2, \dots, \lambda m_d)$, we write $\mathbf{m} \geq \mathbf{n}$ if and only if $m_i \geq n_i$ for $i = 1, 2, \dots, d$, and we write $\mathbf{0}$ for $(0, 0, \dots, 0) \in \mathbb{Z}^d$ and $\mathbf{1}$ for $(1, 1, \dots, 1) \in \mathbb{Z}^d$.

Theorem 1.1. — *Let $A : \mathbb{Z}^d_{\geq 0} \to \mathbb{Z}_p \setminus \{0\}$ and $g : \mathbb{Z}^d_{\geq 0} \to \mathbb{Z}_p \setminus \{0\}$ be maps satisfying the following three properties:*

(*i*) $v_p\big(A(\mathbf{0})\big) = 0$;
(*ii*) $A(\mathbf{n}) \in g(\mathbf{n})\mathbb{Z}_p$;

(iii) for all non-negative integers s and all integer vectors $\mathbf{v}, \mathbf{u}, \mathbf{n} \in \mathbb{Z}_{\geq 0}^d$ with $0 \leq v_i < p$ and $0 \leq u_i < p^s$, $i = 1, 2, \ldots, d$,

$$\frac{A(\mathbf{v} + p\mathbf{u} + \mathbf{n}p^{s+1})}{A(\mathbf{v} + p\mathbf{u})} - \frac{A(\mathbf{u} + \mathbf{n}p^s)}{A(\mathbf{u})} \in p^{s+1} \frac{g(\mathbf{n})}{g(\mathbf{v} + p\mathbf{u})} \mathbb{Z}_p.$$

Then, for all non-negative integers s and all integer vectors $\mathbf{m}, \mathbf{K}, \mathbf{a} \in \mathbb{Z}_{\geq 0}^d$ with $0 \leq a_i < p$, $i = 1, 2, \ldots, d$, we have

$$\sum_{p^s\mathbf{m} \leq \mathbf{k} \leq p^s(\mathbf{m}+\mathbf{1})-\mathbf{1}} \big(A(\mathbf{a} + p\mathbf{k})A(\mathbf{K} - \mathbf{k}) - A(\mathbf{a} + p(\mathbf{K} - \mathbf{k}))A(\mathbf{k})\big) \in p^{s+1} g(\mathbf{m})\mathbb{Z}_p,$$

where we extend A to $\mathbb{Z}^d$ by $A(\mathbf{n}) = 0$ if there is an i such that $n_i < 0$.

While the proofs of Lemmas 2.1 and 6.1 (corresponding to items (D2) and (D4)) are relatively straightforward extensions of the one-variable proofs given in [**17**, Ch. 14, p. 76] and [**10**, proof of Lemma 4.2], respectively, the proofs of Lemma 2.3 and Theorem 1.1 (corresponding to items (D3) and (D5)) need new ideas. The proof of Lemma 2.1 is given in Section 3. Section 5 is devoted to the proof of Lemma 2.3. This proof was kindly provided by an anonymous referee. (For our original proof, see [**16**, Sec. 5].) Even in the one-dimensional case, this proof (as well as the one in [**16**]) is new, as it simplifies the earlier proofs [**15**, proofs of Lemma 1, respectively Lemma 5] and [**14**, proof of Lemma 3]. In fact, it turned out that these earlier proofs could not be extended to the multivariate case. The proof of Lemma 6.1 can be found in Section 6. Finally, in Section 7 we prove Theorem 1.1.

The main application of our multivariate theory of formal congruences that we present in this paper concerns the proof that, for a large class of multivariable mirror maps, their Taylor coefficients are integers. We state the corresponding general theorem in the next subsection. The subsequent subsection collects some particularly interesting special cases and consequences.

1.2. A family of GKZ functions and their associated mirror maps. — In order to state the results in this section conveniently, we need to further enlarge our set of multi-index notations given before Theorem 1.1. Given vectors $\mathbf{m} = (m_1, m_2, \ldots, m_d)$ and $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ in $\mathbb{R}^d$, we write $\mathbf{m} \cdot \mathbf{n}$ for the scalar product $m_1n_1 + m_2n_2 + \cdots + m_dn_d$, and we write $|\mathbf{m}|$ for $m_1 + m_2 + \cdots + m_d$. Furthermore, given a vector $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ of variables and $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$, we write $\mathbf{z}^\mathbf{n}$ for the *product* $z_1^{n_1} z_2^{n_2} \cdots z_d^{n_d}$. On the other hand, if n is an integer, we write $\mathbf{z}^n$ for the *vector* $(z_1^n, z_2^n, \ldots, z_d^n)$.

Given k vectors $\mathbf{N}^{(j)} = (N_1^{(j)}, N_2^{(j)}, \ldots, N_d^{(j)}) \in \mathbb{Z}^d$, $j = 1, \ldots, k$, with $\mathbf{N}^{(j)} \geq \mathbf{0}$, let us define the series

$$F_\mathbf{N}(\mathbf{z}) = \sum_{\mathbf{m} \geq \mathbf{0}} \mathbf{z}^\mathbf{m} \prod_{j=1}^k \frac{(\mathbf{N}^{(j)} \cdot \mathbf{m})!}{\prod_{i=1}^d m_i!^{N_i^{(j)}}} = \sum_{\mathbf{m} \geq \mathbf{0}} \mathbf{z}^\mathbf{m} \prod_{j=1}^k \frac{\big(\sum_{i=1}^d N_i^{(j)} m_i\big)!}{\prod_{i=1}^d m_i!^{N_i^{(j)}}}.$$