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Andrea Pulita

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Lucia Di Vizio, Tanguy Rivoal, eds.

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A BASIC INTRODUCTION TO DEFORMATION AND CONFLUENCE OF ULTRAMETRIC DIFFERENTIAL AND q-DIFFERENCE EQUATIONS

by

Andrea Pulita

Abstract. — This is a short introduction to the phenomena of *deformation and confluence* of linear differential/difference equations, in the ultrametric context, following the papers [3], [24], [23]. It is the transcription of a talk given at the thematic school on *Théories galoisiennes et arithmétiques des équations différentielles*, 21-25 september 2009, at the C.I.R.M. of Luminy (France). These notes are intended to be comprehensible to non specialists, and especially to the undergraduate students of that school.

***Résumé* (Introduction de base à la déformation et confluence des équations différentielles ultramétriques et aux q-différences)**

Celle ci est une courte introduction aux phénomènes de la *déformation et de la confluence* des équations différentielles et aux différences, dans le cadre ultramétrique, suivant les papiers [3], [24], [23]. C'est la transcription d'un exposé donné lors de l'école thématique sur *Théories galoisiennes et arithmétiques des équations différentielles*, 21-25 septembre 2009, qui a eu lieu au C.I.R.M., Luminy (France). Cet écrit est supposé être compréhensible aux non-spécialistes, spécialement aux étudiants qui ont participé à ce rencontre.

Introduction

The aim of this paper is to give a very simple and quick introduction to the phenomena of deformation and confluence in the ultrametric context. The aim is to explain the contents of the papers [3], [24], and [23]. These notes are not intended to be a general survey on the topic of deformation and confluence.

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The notion of stratification (cf. section 4) is essentially due to A.Grothendieck, P.Berthelot, L.Illusie, N.M.Katz, ... [(1)]. They were mainly interested in finding a substitute of the notion of linear differential equation in characteristic $p > 0$, in order to obtain a good category of coefficients for a p-adic cohomological theory. Indeed, in characteristic 0, stratifications form a category which is equivalent to that of (linear) differential equations (cf. Theorem 4.1). In down to earth terms, the notion of stratification is nothing but the data of the generic Taylor solution of a differential equation (cf. section 4).

In these notes we are going to expose, in the ultrametric context, the definition of a functor called σ-deformation. Roughly speaking the functor is obtained as follows. We consider differential modules defined over a 1-dimensional affinoid X (see section 1). We consider a certain type of differential modules (M, ∇) over X, whose Taylor solutions have "*large convergence*". We then prove that for all automorphism σ of X, sufficiently close to the identity, there exists a canonical semi-linear action of σ on the differential module M, making it a so called σ-module (cf. section 3). We denote by $\sigma^{\mathrm{M}} : \mathrm{M} \xrightarrow{\sim} \mathrm{M}$ this operator. The main property of this action of σ is the following: If $(\mathrm{M}, \sigma^{\mathrm{M}})$ is intended as a σ-difference equation over X (see section 3), then its solutions on a disk coincide with the Taylor solutions of M intended as a differential equation. The operator σ^{M} is canonical in the sense that it commutes with the morphisms between differential modules. We hence have a functor (which is the identity on the morphisms) associating to the differential module (M, ∇) the pair $(\mathrm{M}, \sigma^{\mathrm{M}})$. This functor is called σ-*deformation*. The main point of this construction is the existence of σ^{M}. We deduce it by considering a certain pull back of the stratification attached to (M, ∇). In this sense we define the σ-deformation functor as the composite of the equivalence between differential equations and stratifications with a certain pull-back functor defined on the category of stratification with values on σ-modules (cf. section 6.2.1).

Structure of the paper. — In the first four sections we start by introducing differential equations, σ-modules, (elementary) stratifications, and the equivalence between differential equation and stratification. Section 5 concerns Berkovich spaces. This section is expository, and is useful in order to understand the behavior of the radius of convergence of the Taylor solutions of a differential equation, and hence the (ultrametric) convergence locus of a stratification (cf. section 5.3). Section 5 is not essential for the basic understanding of the rest of the paper. In sections 6 and 7 we introduce the σ-deformation and σ-confluence functors. We recall very roughly the method employed by Y.André and L.Di Vizio (cf. [**3**]) to obtain the σ_q-confluence in the case of p-adic q-differences equations over the so called Robba ring. In section

(1) The notion of stratification as intended in these notes certainly date back to before, as actually Grothendieck affirms (without giving any reference, cf. [**15**]). The terminology *stratification* is not standard, and our notion slightly differs from that of Grothendieck (cf.[**6**]).

6.2 we give an alternative construction of the σ-deformation functor as a certain pull-back of the stratification, and we compare this definition with that of Y.André and L.Di Vizio. As a main goal we obtain the σ-deformation functor for a more general class of automorphisms σ, and for more general classes of domains and of equations (cf. section 6.2.3). Moreover we obtain the analytical dependence of the operators on a parameter that can run on an ultrametric analytic variety (cf. section 6.3). In the context of q-difference equations the analytical dependence of the operator σ_q (acting on the module) with respect to q permits to reproduce the analogous of the q-confluence functor for the roots of unity (cf. section 7.4). Indeed we heuristically look to the category of differential equations as a category "over $q = 1$", and that of q-difference equations as a category "over q", where q is different to a root of unity. The classical confluence functor, as exposed in this paper, associates to a q-difference equation a differential equation having the same Taylor solutions at (one and hence at) all points. This is done using the analytical dependence of σ_q (acting on the module) with respect to q (cf. section 7.3.2). We generalize this construction "over $q = \xi_{p^n}$" where ξ_{p^n} is a p^n-th root of unity instead of "over $q = 1$", by replacing the category of differential equations with a category of mixed objects formed by a q-difference module (q equal to a root of unity) together with a (compatible) differential equation. The last section 8 takes a quick look at the complex analogous. No material of this paper is new with the exception of this last section which is intended to be a very quick introduction to a forthcoming paper.

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1. Affinoids

Let K be a field, together with an ultrametric absolute value $|.|$ for which K is complete.[2] The basic bricks of the ultrametric geometry are the so called K-affinoids. In this paper we consider those of them that are (one dimensional, connected) affinoid sub-spaces of the affine line, defined by a family of conditions $X := \{x \mid |x - c_0| \leq R_0,\ |x - c_i| \geq R_i,\ \forall\ i = 1, \dots, n\}$, where $0 < R_1, \dots, R_n \leq R_0$ are arbitrary real numbers, and $c_0, \dots, c_n \in K$. We often indicate X as

$$X \ := \ \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i)\ , \tag{1.1}$$

where the symbol $\mathrm{D}^+(c_0, R_0) = \{x \mid |x - c_0| \leq R_0\}$ means a *closed* disk, and $\mathrm{D}^-(c_i, R_i) = \{x \mid |x - c_i| < R_i\}$ an *open* disk. The assumption $c_0, \dots, c_n \in K$ is due to some technical reasons. If $(A, |.|)/(K, |.|)$ is a complete valued K-algebra

(2) Notice that the absolute value is possibly trivial. All the statements of this paper work as well over an affinoid over a field K together with the trivial absolute value. The results of section 6.1, and also the part of the theory concerning differential/σ-difference equations over the Robba ring need the absolute value to be non-trivial, because one applies the theory of Christol-Mebkhout.

with respect to a norm extending the absolute value of K, then we denote by $X(A)$ the set of elements in A satisfying the conditions of X. As an example if $R_0 = 1$, $R_1 = \cdots = R_n = 0$, and $c_0 = \cdots = c_n = 0$, then for all complete valued field extension Ω/K one has $X(\Omega) = \mathcal{O}_\Omega$, where $\mathcal{O}_\Omega$ is the ring of integers of Ω. In this sense, analogously to the theory of schemes, X is a functor of the category of complete normed K-algebras with values in the category of sets. By abuse of language we will write $x \in X$ to indicate "$x \in X(\Omega)$ for an unspecified complete valued field extension Ω/K".

Let now $K(T)$ be the fraction field of the ring $K[T]$ of polynomials with coefficients in K. The sub-ring $\mathcal{H}_K^{\mathrm{rat}}(X)$ of $K(T)$, formed by rational functions without poles on X, has a norm $\|.\|_X$ defined as

$$\|P(T)/Q(T)\|_X \ := \ \sup_{x\in X}|P(x)/Q(x)|\ . \tag{1.2}$$

Remark *1.1*. — *Here $x \in X$ means that x runs into the set of Ω-rational points of X, for an unspecified field Ω (large enough) equipped with an absolute value $|.|_\Omega$ extending that of K. The correct way to express the above definition would be then $\|P(T)/Q(T)\|_X := sup_{\Omega/K} sup_{x\in X(\Omega)}|f(x)|$, where $X(\Omega)$ means the "Ω-rational points of X". This formulation is possible thanks to the fact that there exists a specific $\widetilde{\Omega}/K$, together with $n+1$ points $t_{c_0,R_0},\ldots,t_{c_n,R_n} \in X(\widetilde{\Omega})$, such that for each other Ω/K one has*

$$\sup_{x\in X(\Omega)}\Big|\frac{P(x)}{Q(x)}\Big|_\Omega \ \leq \ \max_{i=0,\ldots,n}\Big|\frac{P(t_{c_i,R_i})}{Q(t_{c_i,R_i})}\Big|_{\widetilde{\Omega}} \ = \ \sup_{x\in X(\widetilde{\Omega})}\Big|\frac{P(x)}{Q(x)}\Big|_{\widetilde{\Omega}}\ . \tag{1.3}$$

The family $\{t_{c_0,R_0},\ldots,t_{c_n,R_n}\}$ is known as the (Dwork's generic points attached to the) Shilow boundary of X (cf. section 5.1). It is hence enough to consider a single field $\widetilde{\Omega}$. But we will often drop the Ω in the notations, as in the equation (1.2). Analogously, when we say that the poles of P/Q (that are algebraic over K) are not in X, we mean that there are no poles of P/Q neither in $X(K^{\mathrm{alg}})$ nor in $X(\Omega)$ for all Ω/K.

The completion $(\mathcal{H}_K(X), \|.\|_X)$ of $(\mathcal{H}_K^{\mathrm{rat}}(X), \|.\|_X)$ is called the ring of *analytic functions* over X (often called *Krasner's analytic elements* over X). If X is reduced to a closed disc $\mathrm{D}^+(c_0, R_0)$ the elements of $\mathcal{H}_K(X)$ can be expressed as power series $f = \sum_{n\geq 0} a_n(T-c_0)^n$, with $a_n \in K$, converging on $\mathrm{D}^+(c_0, R_0)$. In this case the condition of convergence becomes $\lim_{n\to\infty} |a_n|R_0^n = 0$. Indeed, in the ultrametric world, a series of elements in K converges if and only if its general term tends to 0.

More generally we define the ring of analytic functions over an *open* disk $\mathrm{D}^-(c, R)$, $c \in K$, $R > 0$ as the intersection $\mathcal{A}_K(c, R) := \cap_{R'<R}\mathcal{H}_K(\mathrm{D}^+(c, R'))$. In other words, the elements of $\mathcal{A}_K(c, R)$ are power series $\sum_{n\geq 0} a_n(T-c)^n$ verifying $\lim_n |a_n|(R')^n = 0$, for all $R' < R$.