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ON THE GALOIS GROUPS OF FAMILIES OF REGULAR SINGULAR DIFFERENCE SYSTEMS

by

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Abstract. — We investigate the variation of the dimension of the Galois groups of families of regular singular difference systems using analytic tools.

Résumé **(Sur les groupes de Galois des familles de systèmes aux différences réguliers singuliers)**
Nous étudions la variation de la dimension des groupes de Galois de familles de systèmes aux différences singuliers réguliers au moyen d'outils analytiques.

1. Introduction-Organization

1.1. Introduction-Main results. — In the whole paper, x will denote a *complex* variable and τ (resp. δ) will denote the difference operator acting on a function Y of the complex variable x by $\tau Y(x) = Y(x-1)$ (resp. $\delta Y(x) = (x-1)(Y(x)-Y(x-1))$).

Let us consider :

$$(\mathcal{S}_h): \quad \tau Y = A_h Y, \ A_h \in \mathrm{GL}_n(\mathbb{C}(x,h))$$

a family of regular singular difference systems parameterized by $h \in \mathbb{C} \setminus \Sigma$, Σ being a finite subset of $\mathbb{C}$, and let us denote by G_h the corresponding difference Galois groups over $\mathbb{C}(x)$ (see [11]). Note that, in what follows, the algebraic dependence on h is not essential and could have been replaced by analytic dependence on h on some open subset of $\mathbb{C}$.

In this paper we study the variation of $\dim G_h$ (dimension of the complex linear algebraic group G_h) with respect to h via an analytic approach.

Let us recall that $(\mathcal{S}_h)$ is Fuchsian if $A_h(\infty) = \mathrm{I}_n$, in which case we set $A_{h;\infty} = \lim_{x\to\infty}(x-1)(\mathrm{I}_n - A_h(x)) \in \mathrm{M}_n(\mathbb{C})$. It is moreover nonresonant if,

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for any pair (λ, μ) of distinct eigenvalues of $A_{h;\infty}$, we have $\lambda - \mu \notin \mathbb{Z}$. The system $(\mathcal{S}_h)$ is regular singular if there exists $F_h \in \mathrm{GL}_n(\mathbb{C}(x))$ such that the system defined by $F_h[A_h] := (F_h(x-1))^{-1}A_h(x)F_h(x)$ is Fuchsian. For details about these classical notions, we refer for instance to sections 1 to 4 of [14] and to chapter 9 of [11] and to the references therein.

The regular singular difference systems are classified by their Birkhoff connection matrices; this is in some sense similar to the classification of the regular singular differential systems by means of their monodromy representations. For any $h \in \mathbb{C} \setminus \Sigma$, we associate to $(\mathcal{S}_h)$ its Birkhoff connection matrix $P_h \in \mathrm{GL}_n(\mathbb{C}(\mathbf{x}))$ with $\mathbf{x} = e^{2\pi i x}$ (see section 2). These give rise, for any $h \in \mathbb{C} \setminus \Sigma$, to a family of Galoisian morphisms $\Lambda_h(a,b) := (P_h(a))^{-1}P_h(b) \in G_h$ parameterized by all $(a,b) \in \mathbb{C}^2$ such that $P_h(a)$ and $P_h(b)$ are defined and invertible (this was pointed out for the first time by P. Etingof in [7] in the case of regular q-difference systems and extended to regular singular (q-)difference systems by M. Van der Put and M. Singer in [11]; for a different, more "analytic", point of view in the q-difference case we refer to the work of J. Sauloy in [16]). These morphisms allow us to give a group-theoretic description of the Galois groups G_h -see Theorem 3.2 in section 3.1- (the fact that they generate Zariski-dense subgroups of the Galois groups is proved in [11]) and of their Lie algebras $\mathfrak{g}_h$ -see Theorem 3.3 in section 3.2-.

Using the above-mentioned description of $\mathfrak{g}_h$, we prove, in section 5, under the hypotheses 1. to 3. stated in section 4.1, that :

Theorem. — *Let* $\kappa = \max_{h \in \mathbb{C}\setminus\Sigma} \dim(G_h)$. *Then* $\Theta = \{h \in \mathbb{C} \setminus \Sigma \mid \dim(G_h) = \kappa\}$ *is an open subset of* $\mathbb{C} \setminus \Sigma$ *with discrete complement.*

The following result is an immediate consequence of the above Theorem.

Corollary. — *Suppose that there exists* $h \in \mathbb{C} \setminus \Sigma$ *such that* $G_h = \mathrm{GL}_n(\mathbb{C})$. *Then for any* $h \in \mathbb{C} \setminus \Sigma$ *but, maybe, a discrete subset, we have* $G_h = \mathrm{GL}_n(\mathbb{C})$.

Note that, replacing x by $\frac{x}{h}$, we can make the parameter h be also the step of the equation. We leave the corresponding statements to the reader.

In a different context, the idea of an analytic approach for the study of the variation of Galois groups appears in the work of J. Sauloy in [15, 16] and is an essential motivation for A. Duval and the author's papers [4, 5, 6, 14]; see also L. Di Vizio and C. Zhang's paper [2]. The reader will find more informations about the algebraic meaning of the analytic theory of (q-)difference equations in the works of P. Etingof [7], of J.-P. Ramis and J. Sauloy [12, 13], of J. Sauloy [16] and of M. Singer and M. Van der Put [11]. Moreover, for problems and results related to the main subject of the present paper, we refer the reader to Y. André's paper [1]. Concerning parameterized q-difference equations, we also refer to section 5 of C. Hardouin and M. Singer's paper [8].

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1.2. Organization. — In section 2 we recall useful properties of the regular singular difference systems. In section 3 we give a group-theoretic description of the Galois group of a given regular singular difference system and of its Lie algebra in terms of a corresponding Birkhoff matrix. In section 4 we consider a family of regular singular difference systems parameterized by $h \in \mathbb{C} \setminus \Sigma$ and we study the dependence of corresponding Birkhoff matrices on the parameter h. In section 5 we prove our main theorem concerning the variation of the dimension of the Galois groups.

2. Regular singular systems : a reminder

2.1. Factorial series. — For the material presented in this section, we refer to section 2.1. of [3] and to [10].

A function a defined and holomorphic on some open subset of $\mathbb{C}$ containing some half-plane $\Pi^+_M := \{x \in \mathbb{C} \mid \Re(x) > M\}$ is expandable into a factorial series on Π^+_M if a admits an expansion, convergent on Π^+_M, of the form :

$$\sum_{s=0}^{+\infty} a_s x^{-[s]}$$

where, for all $s \in \mathbb{N}$,

$$x^{-[s]} = \frac{1}{x(x+1)\cdots(x+s-1)}.$$

When it exists, the factorial series expansion is unique. For later use, for all $s \in \mathbb{N}$, we also introduce the following notation :

$$x^{[s]} = x(x+1)\cdots(x+s-1).$$

The set of germs of holomorphic functions expandable into factorial series, denoted by $\mathcal{O}_{fact}$, is by definition the direct limit of the sets of holomorphic functions expandable into factorial series on the half plane Π^+_M as M tends to $+\infty$ (in what follows, we will identify an element of the direct limit with one of its representatives). It is a subring of the ring of germs of holomorphic functions at $+\infty$ which is, by definition, the direct limit of the rings of functions holomorphic on the half plane Π^+_M as M tends to $+\infty$. In particular $\mathcal{O}_{fact}$ is an integral domain; its field of fractions is denoted by $\mathcal{M}_{fact}$. The intersection of $\mathcal{O}_{fact}$ and $\mathcal{M}_{fact}$ with $\mathcal{M}(\mathbb{C})$, the field of meromorphic functions over $\mathbb{C}$, are respectively denoted by $\mathcal{O}_{fact}(\mathbb{C})$ and $\mathcal{M}_{fact}(\mathbb{C})$.

Replacing x by $-x$, we get the notion of function expandable into retrofactorial series. More explicitly, a function a holomorphic on some open subset of $\mathbb{C}$ containing some half-plane $\Pi_M^- := \{x \in \mathbb{C} \mid \Re(x) < M\}$ is expandable into a retrofactorial series on Π_M^- if a admits an expansion, convergent on Π_M^-, of the form :

$$\sum_{s=0}^{+\infty} a_s x^{-[-s]}$$

where, for all $s \in \mathbb{N}$,

$$x^{-[-s]} = \frac{1}{x(x-1)\cdots(x-s+1)}.$$

When it exists, the retrofactorial series expansion is unique. For later use, for all $s \in \mathbb{N}$, we introduce the notation :

$$x^{[-s]} = x(x-1)\cdots(x-s+1).$$

Moreover, we introduce the rings and fields of retrofactorial series $\mathcal{O}_{retrofact}$, $\mathcal{M}_{retrofact}$, $\mathcal{O}_{retrofact}(\mathbb{C})$ and $\mathcal{M}_{retrofact}(\mathbb{C})$ defined similarly to $\mathcal{O}_{fact}$, $\mathcal{M}_{fact}$, $\mathcal{O}_{fact}(\mathbb{C})$ and $\mathcal{M}_{fact}(\mathbb{C})$ respectively.

For instance, any function defined and analytic in a neighborhood of $\infty \in \mathbb{P}^1_{\mathbb{C}}$ is expandable into factorial series and retrofactorial series.

We will denote by $\widehat{\mathcal{O}}_{fact}$ the integral domain of formal factorial series and we denote by $\widehat{\mathcal{M}}_{fact}$ its field of fractions. The ring laws on $\widehat{\mathcal{O}}_{fact}$ are given, for all $a(x) = \sum_{s=0}^{+\infty} a_s x^{-[s]} \in \widehat{\mathcal{O}}_{fact}$ and $b(x) = \sum_{s=0}^{+\infty} b_s x^{-[s]} \in \widehat{\mathcal{O}}_{fact}$, by :

$$(a+b)(x) = \sum_{s=0}^{+\infty} (a_s + b_s) x^{-[s]} \tag{1}$$

and :

$$(ab)(x) = \sum_{s=0}^{+\infty} c_s x^{-[s]} \tag{2}$$

where :

$$c_0 = a_0 b_0 \text{ and, } \forall s \in \mathbb{N}^*, \quad c_s = a_0 b_s + a_s b_0 + \sum_{(j,k,l) \in J_s} c_{j,l}^{(k)} a_j b_l$$

with :

$$\forall s \in \mathbb{N}^*, \quad J_s = \{(j,k,l) \mid j,l \geq 1,\ k \geq 0,\ j+k+l = s\} \tag{3}$$

and :

$$\forall (j,l) \in \mathbb{N}^* \times \mathbb{N}^*,\ \forall\, k \in \mathbb{N}, \quad c_{j,l}^{(k)} = \frac{(j+k-1)!(l+k-1)!}{k!(j-1)!(l-1)!}. \tag{4}$$

As above, replacing x by $-x$, we get the integral domain of formal retrofactorial series $\widehat{\mathcal{O}}_{retrofact}$; its field of fractions is denoted by $\widehat{\mathcal{M}}_{retrofact}$.

We can interpret any element A of $M_{n,m}(\mathcal{O}_{fact})$ or $M_{n,m}(\widehat{\mathcal{O}}_{fact})$ as a series $\sum_{s=0}^{+\infty} A_s x^{-[s]}$ with coefficients in $M_{n,m}(\mathbb{C})$. The above sum and product formulas