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V. P. Spiridonov

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Lucia Di Vizio, Tanguy Rivoal, eds.

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ELLIPTIC HYPERGEOMETRIC TERMS

by

V. P. Spiridonov

Abstract. — General structure of the multivariate plain and q-hypergeometric terms and univariate elliptic hypergeometric terms is described. Some explicit examples of the totally elliptic hypergeometric terms leading to multidimensional integrals on root systems, either computable or obeying non-trivial symmetry transformations, are presented.

***Résumé* (Termes elliptiques hypergÃľomÃľtriques).** — Nous décrivons la structure générale des termes hypergéométriques usuels et q-hypergéométriques de plusieurs variables, ainsi que le terme hypergéométrique elliptique d'une variable. Nous présentons des exemples de termes hypergéométriques totalement elliptiques conduisant à des intégrales multidimensionnelles sur des systèmes de racines, qui soit sont calculables soit obéisssent à des symétries non triviales.

1. Plain hypergeometric case

The definition of the hypergeometric series goes as far back as to Euler and, in a more general setting, to Pochhammer and Horn [1, 7].

Definition 1. — *The formal series*

$$\sum_{m\in\mathbb{Z}^n} c(m) = \sum_{m\in\mathbb{Z}^n} c(m_1,\ldots,m_n)$$

is called plain hypergeometric series, if the ratios

$$\frac{c(m_1,\ldots,m_i+1,\ldots,m_n)}{c(m_1,\ldots,m_n)} = R_i(m_1,\ldots,m_n)$$

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are rational functions of $m_1, \ldots, m_n$.

Suppose that given rational functions $R_i(m)$, called certificates, satisfy the consistency conditions

$$R_i(m_1, \ldots, m_k+1, \ldots, m_n)R_k(m) = R_k(m_1, \ldots, m_i+1, \ldots, m_n)R_i(m).$$

The general form of corresponding (admissible) plain hypergeometric series was determined by Ore and Sato (e.g., see the survey [7]).

Theorem 1. — *General admissible plain hypergeometric terms* $c(m)$ *have the form*

$$c(m) = \frac{R(m)}{\prod_{j=1}^{K}\Gamma(\epsilon_j(m)+a_j)}\prod_{k=1}^{n} z_k^{m_k},$$

where z_k, a_j *are arbitrary complex parameters,* $K = 0, 1, 2, \ldots$, $\epsilon_j(m) = \sum_{k=1}^{n}\epsilon_{jk}m_k$, $\epsilon_{jk} \in \mathbb{Z}$, $R(m)$ *is some rational function of* $m_1, \ldots, m_n$, *and* $\Gamma(x)$ *is the standard Euler gamma function.*

Using the inversion formula $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$, some of the Γ-functions can be put from the denominator of $c(m)$ to its numerator.

In a similar way one can treat hypergeometric integrals [18].

Definition 2. — *The integrals*

$$\int_D \Delta(x)dx = \int_D \Delta(x_1, \ldots, x_n)dx_1 \ldots dx_n,$$

for some domain of integration $D \in \mathbb{C}^n$, *are called plain hypergeometric integrals, if the ratios*

$$\frac{\Delta(x_1 \ldots, x_i+1, \ldots, x_n)}{\Delta(x_1, \ldots, x_n)} = R_i(x_1, \ldots, x_n)$$

are rational functions of $x_1, \ldots, x_n$.

The general admissible plain hypergeometric terms $\Delta(x)$ have the form

$$\Delta(x) = \varphi(x)R(x)\frac{\prod_{j=1}^{K}\Gamma(\sum_{k=1}^{n}\mu_{jk}x_k+b_j)}{\prod_{j=1}^{M}\Gamma(\sum_{k=1}^{n}\epsilon_{jk}x_k+a_j)}\prod_{k=1}^{n} z_k^{x_k},$$

where z_k, a_j, b_j are arbitrary complex parameters, $K, M = 0, 1, 2, \ldots$, $\mu_{jk}, \epsilon_{jk} \in \mathbb{Z}$, $R(x)$ is some rational function of $x_1, \ldots, x_n$, and $\varphi(x) = \varphi(x_1, \ldots, x_i+1, \ldots, x_n)$ is an arbitrary periodic function. The plain hypergeometric series can be obtained from integrals as sums of residues for particular sequences of poles of $\Delta(x)$.

The $\Gamma(x)$-function can be defined as a special meromorphic solution of the functional equation

$$\Gamma(x+1) = x\Gamma(x),$$

the general solution of which has the form $\varphi(x)\Gamma(x)$, where $\varphi(x+1) = \varphi(x)$ is an arbitrary periodic function.

For $n = 1$, definition 2 yields the Meijer function for the choices $\varphi = 1$, $R = 1$ and D one of the contours 1) $\{-\mathrm{i}\infty, +\mathrm{i}\infty\}$ separating sequences of equidistant poles going to the left and to the right of the complex plane, 2) $\{-\infty - \mathrm{i}A, -\infty + \mathrm{i}B\}$ encircling sequences of poles going to the left (for some choice of the positive constants A and B), 3) $\{+\infty - \mathrm{i}A, +\infty + \mathrm{i}B\}$ encircling sequences of poles going to the right (for some choice of the constants A and B).

It is worth to remark that a limiting form of the plain hypergeometric terms is determined by the system of partial differential equations

$$\frac{1}{\Delta(x)}\frac{\partial \Delta(x)}{\partial x_i} = R_i(x).$$

2. q-hypergeometric case

q-deformations of hypergeometric functions were introduced by Heine a long time ago [1].

Definition 3. — *The formal series*

$$\sum_{m\in\mathbb{Z}^n} c(m) = \sum_{m\in\mathbb{Z}^n} c(m_1,\ldots,m_n)$$

is called q-hypergeometric, if

$$\frac{c(m_1,\ldots,m_i+1,\ldots,m_n)}{c(m_1,\ldots,m_n)} = R_i(q^{m_1},\ldots,q^{m_n})$$

are rational functions of $q^{m_1},\ldots,q^{m_r}$*, where* q *is an arbitrary complex parameter.*

This is a natural extension of the previous definition and it leads [7] to the following theorem.

Theorem 2. — *General admissible q-hypergeometric terms* $c(m)$ *have the form*

$$c(m) = R(q^m)\frac{\prod_{j=1}^K (a_j;q)_{\mu_j(m)}}{\prod_{j=1}^M (b_j;q)_{\epsilon_j(m)}}\prod_{k=1}^n x_k^{m_k},$$

where x_k, a_j, b_j *are arbitrary complex parameters,* $K,M = 0,1,2,\ldots$, $\mu_j(m) = \sum_{k=1}^n \mu_{jk}m_k$ *and* $\epsilon_j(m) = \sum_{k=1}^n \epsilon_{jk}m_k$ *with* $\mu_{jk},\epsilon_{jk}\in\mathbb{Z}$, $R(q^m)$ *is some rational function, and*

$$(x;q)_n := \begin{cases}\prod_{j=0}^{n-1}(1-xq^j), & \text{for } n>0\\ \prod_{j=1}^{-n}(1-xq^{-j})^{-1}, & \text{for } n<0\end{cases}$$

is the q-shifted factorial (or the q-Pochhammer symbol).

Definition 4. — *The integrals*

$$\int_D \Delta(x)dx = \int_D \Delta(x_1,\ldots,x_n)dx_1\ldots dx_n,$$

for some domain of integration $D\in\mathbb{C}^n$*, are called q-hypergeometric, if the ratios*

$$\frac{\Delta(x_1,\ldots,x_i+1,\ldots,x_n)}{\Delta(x_1,\ldots,x_n)} = R_i(q^{x_1},\ldots,q^{x_n})$$

are rational functions of $q^{x_1}, \dots, q^{x_n}$.

Define q-gamma functions as special meromorphic solutions of the finite-difference equation

$$f(u+\omega_1) = (1 - e^{2\pi \mathrm{i} u/\omega_2}) f(u), \tag{2.1}$$

where $\omega_1, \omega_2 \in \mathbb{C}$. Evidently, solutions of this equation are defined modulo multiplication by an arbitrary $\varphi(u+\omega_1) = \varphi(u)$ periodic function. Introducing the variables

$$q := e^{2\pi \mathrm{i}\omega_1/\omega_2}, \qquad z := e^{2\pi \mathrm{i} u/\omega_2},$$

this equation can be replaced by

$$\Gamma_q(qz) = (1-z)\Gamma_q(z).$$

For $|q| < 1$ its particular solution, analytic near the point $z = 0$, is determined by a simple iteration, which yields

$$\Gamma_q(z) = \frac{1}{(z;q)_\infty} = \prod_{j=0}^{\infty} \frac{1}{1 - zq^j}, \quad \Gamma_q(0) = 1.$$

This function can be considered as a q-gamma function for $|q| < 1$. More precisely, the Thomae-Jackson q-gamma function has the form

$$\Gamma(u;q) = (1-q)^{1-u} \frac{(q;q)_\infty}{(q^u;q)_\infty}, \qquad \lim_{q \to 1} \Gamma(u;q) = \Gamma(u).$$

It satisfies the equations

$$\Gamma(u+1;q) = \frac{1-q^u}{1-q}\Gamma(u;q), \qquad \Gamma(u - \frac{2\pi \mathrm{i}}{\log q}; q) = (1-q)^{\frac{2\pi \mathrm{i}}{\log q}} \Gamma(u;q).$$

For $|q| < 1$ the q-Pochhammer symbol can be written as

$$(t;q)_n = \frac{(t;q)_\infty}{(tq^n;q)_\infty}, \qquad n \in \mathbb{Z}.$$

For $|q| = 1$ the equation $f(qz) = (1-z)f(z)$ does not have meromorphic solutions for $z \in \mathbb{C}^*$. In this case it is necessary to consider equation (2.1) and search for its solutions meromorphic in $u \in \mathbb{C}$. The modified q-gamma function

$$\gamma(u;\omega_1,\omega_2) = \exp\left(- \int_{\mathbb{R}+\mathrm{i}0} \frac{e^{ux}}{(1-e^{\omega_1 x})(1-e^{\omega_2 x})} \frac{dx}{x} \right), \tag{2.2}$$

where the contour $\mathbb{R} + \mathrm{i}0$ passes along the real axis turning over the point $x = 0$ from above in an infinitesimal way, solves (2.1) and remains meromorphic for $\omega_1, \omega_2 > 0$ when $|q| = 1$. (2.2) is known as the Barnes-Shintani "double sine" function, the noncompact quantum dilogarithm, or the hyperbolic gamma function (see the survey [14] for relevant references).

We assume that $\mathrm{Re}(\omega_1), \mathrm{Re}(\omega_2) > 0$. Then the integral (2.2) is convergent for $0 < \mathrm{Re}(u) < \mathrm{Re}(\omega_1 + \omega_2)$. Under appropriate restrictions on u and $\omega_{1,2}$, it can be