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DRINFELD A-QUASI-MODULAR FORMS

by

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Abstract. — This paper consists of two parts. In the first part, we give an upper bound for the vanishing order at infinity of a non zero Drinfeld quasi-modular form as a function of its weight and its depth. This multiplicity estimate is optimal up to a logarithm factor and improves a previous estimate obtained by the second author. In the second part, we investigate the structure of the algebra of *almost A-quasi-modular forms* (already introduced by the second author) and of some subalgebras. We define in particular the notion of an *A-modular form* and prove a partial result describing their structure.

***Résumé* (Formes A-quasi-modulaires de Drinfeld).** — Cet article comporte deux parties. Dans la première, nous obtenons une majoration de l'ordre d'annulation à l'infini d'une forme quasi-modulaire de Drinfeld non nulle en fonction de son poids et de sa profondeur. Ce lemme de multiplicité est optimal à un facteur logarithme près et améliore une estimation précédente obtenue par le second auteur. Dans la seconde partie, nous étudions la structure de l'algèbre des *A-formes presque quasi-modulaires* (déjà introduites par le deuxième auteur) et celle de quelques sous-algèbres. Nous définissons en particulier la notion de *A-forme modulaire* et prouvons un résultat partiel décrivant leur structure.

1. Introduction

The aim of this article is twofold: first, improve the multiplicity estimate obtained by the second author in [10] for Drinfeld quasi-modular forms; and then, study the structure of certain algebras of *almost-A-quasi-modular forms*, which already appeared in [10].

In order to motivate and describe more precisely our results, let us introduce some notation. Let $q = p^e$ be a power of a prime number p with $e > 0$ an integer, let $\mathbb{F}_q$ be the finite field with q elements. Let θ be an indeterminate over $\mathbb{F}_q$, and write

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$A=\mathbb{F}_q[\theta]$, $K=\mathbb{F}_q(\theta)$. Let $|.|$ be the absolute value on K defined by $|x|=q^{\deg_\theta x}$ for $x\neq 0$, and denote by $K_\infty=\mathbb{F}_q((1/\theta))$ the completion of K with respect to $|.|$, by K_∞^{alg} an algebraic closure of K_∞, and by C the completion of K_∞^{alg} for the unique extension of $|.|$ to K_∞^{alg}.

Let us denote by Ω the rigid analytic space $C\setminus K_\infty$ and by $\Gamma:=\mathbf{GL}_2(A)$ the group of 2×2-matrices with determinant in $\mathbb{F}_q^*$, having coefficients in A. The group Γ acts on Ω by homographies. In this setting, we can define *Drinfeld modular forms* and *Drinfeld quasi-modular forms* for Γ in the usual way (see [3] or Section 2 below for a definition). One of the problems considered in this paper is to prove a *multiplicity estimate* for Drinfeld quasi-modular forms, that is, an upper bound for the vanishing order at infinity of such forms, as a function of the *weight* and the *depth*. In [4] and [10], the following conjecture is suggested ($\nu_\infty(f)$ denotes the vanishing order of f at infinity, see Section 3 for the definition):

Conjecture 1.1. — *There exists a real number $c(q)>0$ such that, for all non-zero quasi-modular form f of weight w and depth $l\geq 1$, one has*

$$\nu_\infty(f)\leq c(q)\,l(w-l). \tag{1}$$

In fact, it is plausible that we can choose $c(q)=1$ in the bound (1). We refer to [4, § 1] or [10, § 1] for further discussion about this question.

In the classical (complex) case, the analogue of Conjecture 1.1 is actually an easy exercice using the resultant $\mathbf{Res}_{E_2}(f,df/dz)$ in the polynomial ring $\mathbb{C}[E_2,E_4,E_6]$ (here E_{2i} denotes the classical Eisenstein series of weight $2i$).

Thus, a natural idea to attack Conjecture 1.1 is to try to mimic this easy proof. However, as explained in [4, § 1.2] and [10, § 1.1], if we do this we are led to use not only the first derivative of f but also its *higher divided derivatives*, or more precisely the sequence of its *hyperderivatives* D_nf, $n\geq 0$, as defined in [3]. But then, due to the erratic behaviour of the operators D_n, obstacles arise which are not easy to overcome, and this approach appears as unfruitful to solve conjecture 1 (see [4, § 1.2] and [10, § 1] for more details).

Another approach to prove Conjecture 1.1 was carried out in [4]. The idea was here to use a *constructive* method: namely, we have constructed explicit families of *extremal* Drinfeld quasi-modular forms, and, by using a resultant argument as above (the function df/dz being replaced now by a suitable extremal form), we were able to get partial multiplicity estimates in the direction of Conjecture 1.1. Unfortunately, we could not construct enough families of extremal forms to prove a general estimate. Thus, also this approach to conjecture 1 seemed unfruitful.

Recently, a new approach was introduced, this time successfully, in [10] to get a general multiplicity estimate (although not optimal) toward Conjecture 1.1. The result obtained is a bound of the form

$$\nu_\infty(f)\leq c(q)\,l^2w\max\{1,\log_q w\}, \tag{2}$$

where $c(q)$ is explicit and $\log_q$ is the logarithm in base q. Morever, it is also proved in [10] that a bound like (1) holds *if* an extra condition of the form $w > c_0(q)l^{5/2}$ is fulfilled ($c_0(q)$ being explicit).

One of the main results of this paper is an improvement of the bound (2), yielding Conjecture 1.1 "up to a logarithm", namely:

Theorem 1.2. — *There exists a real number $c(q) > 0$ such that the following holds. Let f be a non zero quasi-modular form of weight w and depth $l \geq 1$. Then*

$$\nu_\infty(f) \leq c(q)\, l(w-l) \max\{1, \log_q(w-l)\}.$$

Moreover, one can take $c(q) = 252q(q^2-1)$.

The proof of this result will be given in Section 3. It consists in a refinement of the method used in [10]. Recall that the main idea is to introduce a new indeterminate t as in Anderson's theory of t-motives, and to work with certain *deformations* of Drinfeld quasi-modular forms (called *almost A-quasi-modular forms* in [10]), on which the Frobenius $\tau : x \mapsto x^q$ acts. Roughly speaking, these forms are functions $\Omega \to C[[t]]$ satisfying certain regularity properties, as well as transformation formulas under the action of Γ involving *two factors of automorphy*. The precise definitions require quite long preliminaries: they are collected for convenience in Section 2, which is mostly a review of facts taken from [10].

Section 4 is devoted to the problem of clarifying the structure of almost A-quasi-modular forms. More precisely, let $\mathbb{T}_{>0}$ denote the sub-C-algebra of $C[[t]]$ consisting of series having positive convergence radius, and let $\widetilde{\mathcal{M}}$ denote the $\mathbb{T}_{>0}$-algebra of almost A-quasi-modular forms. As for standard Drinfeld quasi-modular forms, almost A-quasi-modular forms have a depth. Denote by $\mathcal{M}$ the sub-algebra of $\widetilde{\mathcal{M}}$ generated by forms of zero depth. Let E denote the "false" Eisenstein series of weight 2 and type 1 defined in [6]. One can define a particular almost A-quasi-modular form denoted by $\boldsymbol{E}$ (see Section 2.3.2), which is a *deformation* of E. In Section 4, we obtain the following partial description of the structure of the algebra $\widetilde{\mathcal{M}}$ (see Theorem 4.1 for the complete statement):

Theorem 1.3. — *The $\mathbb{T}_{>0}$-algebra $\mathcal{M}$ has dimension 3 and the algebra $\widetilde{\mathcal{M}}$ has dimension 5. Moreover, we have $\widetilde{\mathcal{M}} = \mathcal{M}[E, \boldsymbol{E}]$.*

We conjecture that $\mathcal{M}$ is generated by three elements that can be explicitly given (see Conjecture 4.9). However, we don't know how to prove this yet.

In the very last part of Section 4, we define the notion of *A-modular forms*: they generate a sub-$\mathbb{T}_{>0}$-algebra of $\widetilde{\mathcal{M}}$ denoted by $\mathbb{M}$. We show (Theorem 4.10) that the algebra $\mathbb{M}$ is of finite type and dimension three over $\mathbb{T}_{>0}$ and we determine explicit generators.

2. Preliminaries

This section collects the preliminaries which will be needed in the next two sections. This is essentially a review of the paper [10].

2.1. Drinfeld modular forms and quasi-modular forms.— The now classical theory of *Drinfeld modular forms* started with the work of Goss (see [8]) and was improved by Gekeler (cf. [6]). We recall here briefly the basic definitions and properties of Drinfeld quasi-modular forms. The reader is referred to [3] for more details and proofs.

We will use the notations of the preceding section. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \Omega$, we will denote by $\gamma(z) = \frac{az+b}{cz+d}$ the image of the homographic action of the matrix γ on z. We will further denote by $\tau : x \mapsto x^q$ the Frobenius endomorphism, generator of the skew polynomial ring $C[\tau] = \mathbf{End}_{\mathbb{F}_q-\mathrm{lin.}}(\mathbb{G}_a(C))$.

Let $\Phi_{\mathrm{Car}} : A \to C[\tau]$ be the Carlitz module, defined by

$$\Phi_{\mathrm{Car}}(\theta) = \theta\tau^0 + \tau.$$

Let $\tilde{\pi}$ be one of its fundamental periods (fixed once for all), and let $e_{\mathrm{Car}} : C \to C$ be the associated exponential function. We have $\ker e_{\mathrm{Car}} = \tilde{\pi}A$ and the function e_{Car} has the following entire power series expansion, for all $z \in C$:

$$e_{\mathrm{Car}}(z) = \sum_{i\geq 0} \frac{z^{q^i}}{d_i}, \tag{3}$$

where, borrowing classical notations,

$$d_0 = 1, \quad d_i = [i][i-1]^q \ldots [1]^{q^{i-1}} \text{ for } i \geq 1 \tag{4}$$

and $[i] = \theta^{q^i} - \theta$. We define the parameter at infinity ([1]) u by setting, for $z \in \Omega$,

$$u = u(z) := \frac{1}{e_{\mathrm{Car}}(\tilde{\pi}z)}.$$

We will say that a function $f : \Omega \to C$ is holomorphic on Ω if it is analytic in the rigid analytic sense, and will say that it is *holomorphic at infinity* if it is A-periodic (that is, $f(z+a) = f(z)$ for all $z \in \Omega$ and all $a \in A$) and if there is a real number $\epsilon > 0$ such that, for all $z \in \Omega$ satisfying $|u(z)| < \epsilon$, $f(z)$ is equal to the sum of a convergent series

$$f(z) = \sum_{n\geq 0} f_n u(z)^n,$$

where $f_n \in C$. In the sequel, we will often identify such a function with a formal series in $C[[u]]$, thus simply writing

$$f = \sum_{n\geq 0} f_n u^n.$$

(1) Note that this parameter is sometimes denoted by $t(z)$ in the literature, *e.g.* in [6] and [3].