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AN INTRODUCTION TO MALGRANGE PSEUDOGROUP

by

Guy Casale

Abstract. — The pseudogroup defined by B. Malgrange as a generalization of the differential Galois group for nonlinear differential equation is presented. It is proved that a equation integrable by quadratures has a solvable pseudgroup. From this a new proof of a theorem of M. Singer is given.

***Résumé* (Une introduction au pseudo-groupe de Malgrange).** — Nous donnons une introduction au pseudo-groupe défini par B. Malgrange généralisant aux équations différentielles non linéaires le groupe de Galois différentiel. Nous prouvons que le pseudo-groupe d'une équation intégrable par quadratures est résoluble et donnons une nouvelle preuve d'un théorème de M. Singer.

1. Introduction

Between 1887 and 1904, E. Picard [18] and E. Vessiot [24] applied ideas from Galois theory to differential equations. They succeded in getting a complete theory in case of linear differential equations nowadays known as Picard-Vessiot Theory. Almost at the same times J. Drach [7] and E. Vessiot [25] tried to extend this theory to a Differential Galois Theory involving also nonlinear equations. Two reciprocal pseudogroups are defined in [25, 26] by Vessiot, the specific one and the rationality one. One can find this definition in the introduction (paragraph 3.) of [25]:

> The specific group is the smallest rational group containing the equation as infinitesimal subgroup.

No one follows this direction until two independent articles of H. Umemura [23] and B. Malgrange [15]. H. Umemura infinitesimal Galois group of a differential equation is the rationality pseudogroup of Vessiot defined rigorously by a Lie-Ritt functor. B. Malgrange without knowledge of this late Vessiot's article gives almost the same definition: Galois pseudogroup of a vector field is the smallest algebraic pseudogroup

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containing this vector field as infinitesimal transformation so it is the specific pseudogroup of Vessiot. Note that Lie-Ritt functors are very close to the way used by E. Cartan to study Lie pseudogroups by means of what is now called Cartan connections and are also very closed to 'virtual groups' defined by B. Malgrange in [16].

In the case of linear differential equations, these two groups appear as the Galois group and the intrinsic group of Katz [2, 12] which are respectively the rationality group and the specific group of Vessiot. In special nonlinear cases these two groups appear to have been introduced by A. Pillay in [19] (Isomorphism between Galois group and the kernel of the differential structure of the intrinsic group is given by lemma 3.11). Analogues of these groups, pseudogroups and their isomorphisms are discussed in [6] in more difficult context of linear q-difference equations.

The point of view presented here is the point of view of B. Malgrange with some (very) small modifications. The end of the introduction contains a first look at the linear case and the main tool used : theorem 1.3. We give the Galois/Umemura group and the Malgrange pseudogroup in the linear case and compare them. Justifications are given in the last section. It appears that this two objects are isomorphic but theirs actions are reciprocal as claimed by Vessiot.

In the second section, algebraic singular groupoids and associated objects are defined. Same examples are given after each definition to understand their relationship.

The third section ends with the definition of Malgrange pseudogroup and the projection theorem which is the only remainder of the Galois correspondence at this stage.

In the last section we prove the nonlinear analog of a classical result due to Kolchin in Picard-Vessiot Theory: the Galoisian object attached to an equation integrable by quadratures is solvable. Here the Galoisian object is rather the Lie algebra of Malgrange pseudogroup than the pseudogroup itself mainly because no good enough definition of solvability for pseudogroup is known.

Before first examples let us recall the set theoretical definition of a pseudogroup. A more algebraic definition will be given in the pseudogroup section.

Definition 1.1. — *A pseudogroup of transformations of an analytic manifold V is a set $\mathcal{G}$ of analytic maps between open sets of V, $\varphi : s(\varphi) \to t(\varphi)$ such that*

- *the restriction of a transformation φ of $\mathcal{G}$ to a open subset of its domain $s(\varphi)$ is in $\mathcal{G}$;*
- *if $\psi \in \mathcal{G}$ and $\varphi \in \mathcal{G}$ and $t(\varphi) = s(\psi)$ then $\psi \circ \varphi$ is in $\mathcal{G}$;*
- *if $\varphi \in \mathcal{G}$ then φ is invertible and $\varphi^{\circ -1}$, the inverse for the composition law, is in $\mathcal{G}$;*
- *if $\varphi : s(\varphi) \to t(\varphi)$ is invertible and $U \subset s(\varphi)$ is an open subset such that $\varphi|_U$ is in $\mathcal{G}$ then φ is in $\mathcal{G}$.*

1.1. Constant linear equations. — Let G an algebraic group of dimension d and X a right invariant vector field on G. The description of the Malgrange pseudogroup of X can be done as follow. Let $Y_1, \ldots, Y_d$ be a basis over $\mathbb{C}$ of left invariant vector fields on G and $H_1, \ldots, H_n$ be generators of the field of rational first integrals of X. It is an easy lemma to prove that left invariant vector fields are infinitesimal generators of right translations, see [13]. Because of this, one gets $[X, Y_i] = 0$ for all $1 \leq i \leq d$ and by definition $X \cdot H_j = 0$ for all $1 \leq j \leq n$. One gets (see definition 3.9)

$$(1) \qquad Mal = \Big\{\varphi \text{ map between open sets } s(\varphi) \text{ and } t(\varphi) \text{ of } G \\ \Big| \varphi^* H_j = H_j, \varphi^* Y_i = Y_i; \forall (i,j), 1 \leq j \leq n, 1 \leq i \leq d\Big\}$$

This pseudogroup is easy to define by means of subgroups of G. Let $K \subset G$ the smallest algebraic group such that X belongs to the right Lie algebra of K.

Lemma 1.2. — *Mal is the pseudogroup of analytic localisations of elements of K acting by left translation on G.*

Proof. — For $g \in G$ let rg (resp. lg) be the right (resp. left) translation by g on G. Because left invariant vector fields are infinitesimal generators of right translations one gets from invariance of Y's $\varphi \circ rg = rg \circ \varphi$. Applying to the neutral element e one gets $\varphi(g) = \varphi(e)g$ thus φ is left translation on its domains. Let K be the subgroup of all $k = \varphi(e)$ for all $\varphi \in Mal$. It is an algebraic subgroup defined by $H_i(k) = H_i(e)$ for all i.

If a smaller algebraic subgroup contains X in its Lie algebra then by Chevalley theorem it has more rational invariants. But all rational invariants of such a group are rational first integrals of X. From the beginning we get all such integrals so K is the smallest algebraic group whose Lie algebra contains X. □

Chevalley theorem can be replaced by the following more general theorem.

Theorem 1.3 **(Gomez-Mont [9]).** — *Let $\mathcal{F}$ be an holomorphic foliation of a projective variety V whose leaves are quasiprojective then there is a variety W and a rational map $H : V \dashrightarrow W$ such that the closure of general fibers of H are closure of leaves of $\mathcal{F}$.*

This theorem can replace Chevalley theorem by setting $G = V$, K is the smallest algebraic subgroup whose Lie algebra contains X and $\mathcal{F}$ is the foliation by orbits of left action of K.

1.2. Linear equations. — Let us consider a linear differential equations on the trivial vector bundle $\mathbb{C}_x \times \mathbb{C}_y^n$

$$(E) \qquad \frac{d}{dx}(y_1, \dots, y_n) = (y_1, \dots, y_n)A(x)$$

with $A \in \mathfrak{gl}_n(\mathbb{C}(x))$. For simplicity such system will be written using the vector field

$$\frac{\partial}{\partial x} + yA(x)\frac{\partial}{\partial y}$$

where $y = (y_1, \dots, y_n)$ and $\frac{\partial}{\partial y} = \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}$.

1.2.1. *Going from the equation to its fundamental form.* —

Definition 1.4. — *A general solution is a vector of n holomorphic functions $y(x, c_1, \dots, c_n)$ of $1+n$ arguments such that $\frac{\partial}{\partial x}y = yA(x)$ and* $\det \frac{\partial y}{\partial c} \not\equiv 0$.

Because of linearity, the Jacobian satisfies

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial c}\right) = \left(\frac{\partial y}{\partial c}\right)A(x)$$

Lemma 1.5. — *Up to some change of integrating constants, one can assume that the dependency in c is linear, i.e. $y = c\frac{\partial y}{\partial c}$.*

The matrix $\frac{\partial y}{\partial c}$ is a fundamental solution of equation (E). Such a general solution is called a linear general solution

1.2.2. *GL_n action from the right or the left.* —

Lemma 1.6. — *If $c(d)$ is a invertible transformation such that for any linear general solution $y(x, c)$, $y(x, c(d))$ is another linear general solution then $c(d)$ is a linear transformation i.e. $c(d) = dC$ for a matrix $C \in GL_n(\mathbb{C})$.*

Two linear general solutions with same domain are related by a linear change of c. This lemma is classical: two fundamental solution are related by multiplication on the left by a constant coefficient invertible matrix.

Lemma 1.7. — *If $(x, \overline{y}(x, y))$ is a invertible transformation such that for any linear general solution $y(x, c)$, $\overline{y}(x, y(x, c))$ is a linear general solution then $\overline{y}(x, y)$ is a linear gauge transformation i.e. $\overline{y}(x, y) = yY(x)$ for some open set U of $\mathbb{C}$ and $Y \in GL_n(\mathcal{O}(U))$ satisfying $\frac{\partial}{\partial x}Y = [Y, A(x)]$.*

Two linear general solutions with same range are related by a linear gauge transformation. Let y be linear general solution with range over $U \subset \mathbb{C}$. Then for any $C \in GL_n(\mathbb{C})$ there is a $Y \in GL_n(\mathcal{O}(U))$ solution of $\frac{\partial}{\partial x}Y = [Y, A(x)]$ such that $y(x, dC) = y(x, d)Y$. This correspondence is one to one.