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LECTURES ON HALL ALGEBRAS

by

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Abstract. — This is a survey of the theory of Hall algebras of abelian categories, with a special focus on examples. After providing the definition and giving the basic properties of the Hall algebra of a finitary category, we deal in some details with the three following cases: the Jordan quiver and the classical Hall algebra; quivers and quantum groups; coherent sheaves over smooth projective curves and quantum loop algebras. We finish with a brief chapter on the Drinfeld doubles of Hall algebras and their relationship with derived categories.

Résumé. — Ce texte est un survol de la théorie des algèbres de Hall des catégories abéliennes. Notre approche est essentiellement basée sur l'étude de nombreux exemples. Après avoir donné les définitions et propriétés de base des algèbres de Hall, nous traitons les case suivants: carquois de Jordan et algèbre de Hall classique; carquois et groupes quantiques; courbes projectives lisses et groupes quantiques de lacets. Le dernier chapitre est dédié à la notion de double de Drinfeld d'une algèbre de Hall ainsi qu'aux rapports que ceux-ci entretiennent avec les catégories dérivées.

Introduction

These notes represent the written, expanded and improved version of a series of lectures given at the winter school "Representation theory and related topics" held at the ICTP in Trieste in January 2006, and at the summer school "Geometric methods in representation theory" held at Grenoble in June 2008. The topic for the lectures was "Hall algebras" and I have tried to give a survey of what I believe are the most fundamental results and examples in this area. The material was divided into five sections, each of which initially formed the content of (roughly) one lecture. These are, in order of appearance on the blackboard:

- Lecture 1. Definition and first properties of (Ringel-)Hall algebras,
- Lecture 2. The Jordan quiver and the classical Hall algebra,

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- Lecture 3. Hall algebras of quivers and quantum groups,
- Lecture 4. Hall algebras of curves and quantum loop groups,
- Lecture 5. The Drinfeld double and Hall algebras in the derived setting.

By lack of time, chalk, (and yes, competence!), I was not able to survey with the proper due respect several important results (notably Peng and Xiao's Hall Lie algebra associated to a 2-periodic derived category [?], Kapranov and Toën's versions of Hall algebras for derived categories, see [?, ?], or the recent theory of Hall algebras of cluster categories, see [?], [?], or the recent use of Hall algebra techniques in counting invariants such as in Donaldson-Thomas theory, see [?, ?, ?], ...). These are thus largely absent from these notes. Also missing is the whole geometric theory of Hall algebras, initiated by Lusztig [?]: although crucial for some important applications of Hall algebras (such as the theory of crystal or canonical bases in quantum groups), this theory requires a rather different array of techniques (from algebraic geometry and topology) and I chose not to include it here, but in the companion survey [?]. More generally, I apologize to all those whose work deserves to appear in any reasonable survey on the topic, but is unfortunately not to be found in this one. Luckily, other texts are available, such as [?, ?, ?]. There are essentially no new results in this text.

Let me now describe in a few words the subject of these notes as well as the content of the various lectures.

Roughly speaking, the Hall, or Ringel-Hall algebra $\mathbf{H}_{\mathcal{A}}$ of a (small) abelian category $\mathcal{A}$ encodes the structure of the space of *extensions* between objects in $\mathcal{A}$. In slightly more precise terms, $\mathbf{H}_{\mathcal{A}}$ is defined to be the $\mathbb{C}$-vector space with a basis consisting of symbols $\{[M]\}$, where M runs through the set of isomorphism classes of objects in $\mathcal{A}$; the multiplication between two basis elements $[M]$ and $[N]$ is a linear combination of elements $[P]$, where P runs through the set of extensions of M by N (i.e. middle terms of short exact sequences $0 \to N \to P \to M \to 0$), and the coefficient of $[P]$ in this product is obtained by counting the number of ways in which P may be realized as an extension of M by N (see Lecture 1 for details). Of course, for this counting procedure to make sense $\mathcal{A}$ has to satisfy certain strong finiteness conditions (which are coined under the term *finitary*), but there are still plenty of such abelian categories around. Another fruitful, slightly different (although equivalent) way of thinking about the Hall algebra $\mathbf{H}_{\mathcal{A}}$ is to consider it as the algebra of finitely supported functions on the "moduli space" $\mathcal{M}_{\mathcal{A}}$ of objects of $\mathcal{A}$ (which is nothing but the set of isoclasses of objects of $\mathcal{A}$, equipped with the discrete topology), endowed with a natural convolution algebra structure (this is the point of view that leads to some more geometric versions of Hall algebras, as in [?, ?, ?]).

Thus, whether one likes to think about it in more algebraic or more geometric terms, Hall algebras provide rather subtle invariants of finitary abelian categories. Note that it is somehow the "first order" homological properties of the category $\mathcal{A}$ (i.e. the structure of the groups $\mathrm{Ext}^1(M,N)$) which directly enter the definition of $\mathbf{H}_{\mathcal{A}}$, but $\mathcal{A}$ may a priori be of arbitrary (even infinite) homological dimension. However, as discovered by Green [?], when $\mathcal{A}$ is *hereditary* , i.e. of homological dimension one or less, it is possible to define a comultiplication $\Delta : \mathbf{H}_{\mathcal{A}} \to \mathbf{H}_{\mathcal{A}} \otimes \mathbf{H}_{\mathcal{A}}$ and, as was later realized by Xiao [?], an antipode $S : \mathbf{H}_{\mathcal{A}} \to \mathbf{H}_{\mathcal{A}}$. These three operations are all compatible and endow (after a suitable and harmless twist which we prefer to ignore in this introduction) $\mathbf{H}_{\mathcal{A}}$ with the structure of a Hopf algebra. All these constructions are discussed in details in Lecture 1.

As the reader can well imagine, the above formalism was invented only after some motivating examples were discovered. In fact, the above construction appears in various (dis)guises in domains such as modular or p-adic representation theory (in the form of the functors of parabolic induction/restriction), number theory and automorphic forms (Eisenstein series for function fields), and in the theory of symmetric functions. The first occurence of the concept of a Hall algebra can probably be traced back to the early days of the twentieth century in the work of E. Steinitz (a few years before P. Hall was born) which, in modern language, deals with the case of the category $\mathcal{A}$ of abelian p-groups for p a fixed prime number. This last example, the so-called *classical* Hall algebra is of particular interest due to its close relation to several fundamental objects in mathematics such as symmetric functions (see [?]), flag varieties and nilpotent cones. After studying in some details Steintiz's classical Hall algebra we briefly state some of the other occurences of (examples of) Hall algebras in Lecture 2.

The interest for Hall algebras suddenly exploded after C. Ringel's groundbreaking discovery ([?]) in the early 1990s that the Hall algebra $\mathbf{H}_{\mathrm{Rep}\,\vec{Q}}$ of the category of $\mathbb{F}_q$-representations of a Dynkin quiver $\vec{Q}$ (equiped with an arbitrary orientation) provides a realization of the positive part $\mathbf{U}(\mathfrak{b})$ of the enveloping algebra $\mathbf{U}(\mathfrak{g})$ of the simple complex Lie algebra $\mathfrak{g}$ associated to the same Dynkin diagram (to be more precise, one gets a quantized enveloping algebra $\mathbf{U}_v(\mathfrak{g})$, where the deformation parameter v is related to the order q of the finite field $\mathbb{F}_q$).

It is also at that time that the notion of a Hall algebra associated to a finitary category was formalized (see [?]). These results were subsequently extended to arbitrary quivers in which case one gets (usually infinite-dimensional) Kac-Moody algebras, and were later completed by Green. The existence of a close relationship between the representation theory of quivers on one hand, and the structure of simple or Kac-Moody Lie algebras on the other hand was well-known since the seminal work of Gabriel, Kac and others on the classification of indecomposable representations of quivers (see [?, ?]). Hall algebras thus provide a concrete, beautiful (and useful !) realization of this correspondence. After recalling the forerunning results of Gabriel and Kac, we state and prove Ringel's and Green's fundamental theorems in the third Lecture.

Apart from the categories of $\mathbb{F}_q$-representations of quivers, a large source of finitary categories of global dimension one is provided by the categories $\mathrm{Coh}(X)$ of coherent sheaves on some smooth projective curve X defined over a finite field $\mathbb{F}_q$. As pointed out by Kapranov in [?], the Hall algebra $\mathbf{H}_{\mathrm{Coh}(X)}$ may be interpreted in the context of automorphic forms over the function field of X. Using this interpretation, he wrote down a set of relations satisfied by $\mathbf{H}_{\mathrm{Coh}(X)}$ for an arbitrary X (these relations involve as a main component the zeta function of X). These relations turn out to determine completely $\mathbf{H}_{\mathrm{Coh}(X)}$ when $X \simeq \mathbb{P}^1$ but this is most likely not true in higher genus (see [?], however, for a combinatorial approach).

In another direction, H. Lenzing discovered in the mid-80's some important generalizations $\mathrm{Coh}(\mathbb{X}_{p,\lambda})$ of the category $\mathrm{Coh}(\mathbb{P}^1)$– the so-called weighted projective lines– which depend on the choice of points $\lambda_1, \ldots, \lambda_r \in \mathbb{P}^1$ and multiplicities $p_1, \ldots, p_r \in \mathbb{N}$ associated to each point ([?]). The category $\mathrm{Coh}(\mathbb{X}_{p,\lambda})$ is hereditary and shares many properties with the categories $\mathrm{Coh}(X)$ of coherent sheaves on curves (not necessarily of genus zero). In fact, in good characteristics, $\mathrm{Coh}(\mathbb{X}_{p,\lambda})$ is equivalent to the category of *G-equivariant* coherent sheaves on some curve Y acted upon by a finite group G, for which $Y/G \simeq \mathbb{P}^1$. The Hall algebras $\mathbf{H}_{\mathrm{Coh}(\mathbb{X}_{p,\lambda})}$ are studied in [?], where it is shown that they provide a realization of the positive part of quantized enveloping algebras of *loop algebras* of Kac-Moody algebras. Note that these algebras are in general not Kac-Moody algebras: for instance when $\mathbb{X}_{p,\lambda}$ is of "genus one" one gets the double affine, or elliptic Lie algebras $\mathcal{E}_{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}, s^{\pm 1}] \oplus \mathbb{K}$ for a Lie algebra $\mathfrak{g}$ of type D_4, E_6, E_7 or E_8. Simultaneously, Crawley-Boevey was led in his beautiful work on the Deligne-Simpson problem [?] to study the classes of indecomposable sheaves in $\mathrm{Coh}(\mathbb{X}_{p,\lambda})$ and found them to be related to loop algebras of Kac-Moody algebras as well (see [?]). The above results concerning Hall algebras of coherent sheaves on curves form the content of Lecture 4, and should be viewed as analogues, in the context of curves, of Gabriel's, Kac's and Ringel's theorems for quivers.

Finally in the last lecture, we state various results and conjectures regarding the behavior of Hall algebras under *derived* equivalences. Recall that taking the Drinfeld double is a process which turns a Hopf algebra $\mathbf{H}$ into another one $\mathbf{DH}$ which is twice as big as $\mathbf{H}$ and which is self-dual; for instance the Drinfeld double of the positive part $\mathbf{U}_v(\mathfrak{b})$ of a quantized enveloping algebra is isomorphic to the whole quantized enveloping algebra $\mathbf{U}_v(\mathfrak{g})$. The guiding heuristic principle –which has recently been established in a wide class of cases by T. Cramer [?]– is that although the Hall algebras $\mathbf{H}_{\mathcal{A}}$ and $\mathbf{H}_{\mathcal{B}}$ of two derived equivalent finitary *hereditary* categories need not be isomorphic, their Drinfeld doubles $\mathbf{DH}_{\mathcal{A}}$ and $\mathbf{DH}_{\mathcal{B}}$ should be. More generally, any fully faithful triangulated functor $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ between derived categories should give rise to a homomorphism of algebras $F_\star : \mathbf{DH}_{\mathcal{A}} \to \mathbf{DH}_{\mathcal{B}}$. In particular, the group of autoequivalences of the derived category $D^b(\mathcal{A})$ is expected to act on $\mathbf{DH}_{\mathcal{A}}$ by algebra automorphisms. As supporting example and motivation for the above principle, we show how the group $Aut(D^b(\mathrm{Coh}(X)))$ for an elliptic curve X acts on $\mathbf{DH}_{\mathrm{Coh}(X)}$. This action turns out to be the key point in understanding the structure of the algebra $\mathbf{DH}_{\mathrm{Coh}(X)}$ (the *elliptic Hall algebra* studied in [?]).

A recent theorem of Happel [?] states that any (connected) hereditary category which is linear over an algebraically closed field k and which possesses a tilting object (see Lecture 5.) is derived equivalent to either $\mathrm{Rep}_k \vec{Q}$ for some quiver $\vec{Q}$ or $\mathrm{Coh}(\mathbb{X}_{p,\lambda})$ for some weighted projective line $\mathbb{X}_{p,\lambda}$. Although the case of categories which are linear over a finite field k is slightly more complicated (see [?], and also [?]), if one believes the above heuristic principle then the results of Lectures 3 and 4 essentially describe the Hall algebra of any finitary hereditary category which possesses a tilting object. Of course the case of finitary hereditary categories which *do not* possess a tilting object (this corresponds to curves of higher genus) is still very mysterious, as is the case of categories of higher global dimension (this corresponds to higher-dimensional varieties) for which virtually nothing is known.

A final word concerning the style of these Lecture notes. They follow a leisurely pace and many examples are included and worked out in details. Nevertheless, because they are mostly (though not only!) aimed at people interested in representation theory of finite-dimensional algebras, I have decided to assume some basic homological algebra and, starting from Lecture 3, a little familiarity with quivers. On the other hand, I assume nothing from Lie algebras and quantum groups theory. Hence I have included in a long appendix a "crash course" on simple and Kac-Moody Lie algebras, loop algebras, and the corresponding quantum groups.

The first four Lectures follow each other in a logical order, but a reader allergic to examples could well jump to Lecture 5 directly after Lecture 1.

1. Lecture 1

The aim of this first Lecture is to introduce in as much generality as possible the notion of the Hall algebra of a finitray abelian category, and to describe in details all the extra structures (coproduct, antipode,...) which have been discovered over the time and which one can put on such an algebra. A final paragraph briefly discusses some functoriality properties of this construction. Examples of Hall algebras abound in Lectures 2, 3 and 4, and the reader is invited to have a look at them as he proceeds through this first Lecture.

1.1. Finitary categories

1.1.1. — A small abelian category $\mathcal{A}$ is called *finitary* if the following two conditions are satisfied:

i) For any two objects $M, N \in \mathrm{Ob}(\mathcal{A})$ we have $|\mathrm{Hom}(M, N)| < \infty$,
ii) For any two objects $M, N \in \mathrm{Ob}(\mathcal{A})$ we have $|\mathrm{Ext}^1(M, N)| < \infty$.

In most, if not all examples of finitary categories which we will be considering in these notes, $\mathcal{A}$ is linear over some finite field $\mathbb{F}_q$, and we have

$$\dim \mathrm{Hom}(M, N) < \infty, \qquad \dim \mathrm{Ext}^1(M, N) < \infty \tag{1.1}$$

for any pair of objects $M, N \in \mathrm{Ob}(\mathcal{A})$. Examples of such categories are provided by the categories $\mathrm{Rep}_{\mathbb{F}_q} \vec{Q}$ of (finite dimensional) $\mathbb{F}_q$-representations of a quiver, or more generally by the categories $\mathrm{Mod}\, A$ of finite-dimensional representations of a finite-dimensional $\mathbb{F}_q$-algebra A. For another class of examples of a more geometric flavor, one may consider the categories $\mathrm{Coh}(X)$ of coherent sheaves on some projective