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CLUSTER CATEGORIES, m-CLUSTER CATEGORIES AND DIAGONALS IN POLYGONS

by

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Abstract. — The goals of this expository article are on one hand to describe how to construct (m-) cluster categories from triangulations (resp. from (m+2)-angulations) of polygons. On the other hand, we explain how to use translation quivers and their powers to obtain the m-cluster categories directly from the diagonals of a polygon.

Résumé **(Catégories de clusters et de m-clusters, et diagonales dans des polygones)**
Le but de cet article d'exposition est d'une part de décrire des constructions des catégories de clusters (amassées) et de m-clusters, à partir de triangulations et de (m+2)-angulations de polygones. D'autre part, nous expliquons comment utiliser des carquois de translations et leur puissances pour obtenir directement les catégories de m-clusters à partir des diagonales d'un polygone.

Introduction

This expository article is the expanded version of a talk given at the conference at the Grenoble summer school "Geometric methods in representation theory" in July 2008. The goal of this talk was to explain how cluster categories and m-cluster categories can be described via diagonals and so-called m-diagonals in a polygon. And then how the latter can actually be described using the power of a translation quiver. The first section gives a very brief introduction to the theory of cluster algebras and cluster categories. It also introduces the notations used in the article. In Section 2, we explain the notions of a quiver given by the diagonals in a polygon and of the one given by m-diagonals. The results in this section are mainly due to Caldero-Chapoton-Schiffler ([6]), to Schiffler ([21]) and to Baur-Marsh ([2], [1]). In the last section, we introduce the concept of the power of a translation quiver. Here, the results are from [2], [1] and from the masters thesis of C. Ducrest ([7]).

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1. Cluster algebras and cluster categories

Cluster algebras were introduced by Fomin and Zelevinsky ([12]) in order to provide an algebraic framework for the phenomena of total positivity and for the canonical bases of the quantized universal enveloping algebras.

We briefly illustrate the notion of total positivity: An $n\times n$ matrix is called *totally positive* if all its minors are positive. Originally, the term was used to describe matrices with non-negative minors: these matrices are nowadays called *totally non-negative*. In the 1930s, Gantmacher-Krein and I. Schoenberg have independently started investigating such matrices. One of the motivations was to estimate the number of real zeroes of a polynomial.

Gantmacher showed that totally non-negative matrices have different real eigenvalues. The interest in total positivity was renewed in the 1990s when G. Lusztig extended the notion to reductive algebraic groups, cf. [18].

Example 1.1. — To illustrate the notion on a (non-reductive) example, let us consider the group of 3 by 3-matrices with 1's on the diagonal and zeroes below. If $U = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ is such a matrix then U is totally positive if $a, b, c > 0$ and $ac - b > 0$. One can check that it is actually enough to require $a > 0$, $ac - b > 0$ and $b > 0$: the condition $c > 0$ will follow automatically. Equivalently, the condition $a > 0$ can be dropped and is automatically satisfied by the remaining conditions. So there is only a certain number of minors that need to be checked.

Furthermore, if from the set $\{a > 0, ac - b > 0, b > 0\}$ of conditions the first is omitted then we may replace it exactly with one other condition, namely with $c > 0$, to obtain total positivity of the matrix.

More generally, one shows that the minimal sets of minors to check all have the same cardinality. And it is often the case that if you remove one minor from such a minimal set, there exists exactly one other minor to replace it with.

1.1. Cluster algebras. — A cluster algebra $\mathcal{A} \subset \mathbb{Q}(u_1, \dots, u_n)$ of rank n is an algebra with possibly infinitely many generators. These generators are called cluster variables; they are arranged in overlaping sets of the same cardinality n, the *clusters*. There are relations between the cluster variables, encoded in an $n \times n$ matrix, the *mutation matrix*. Through mutation, one element of a cluster is exchanged by exactly one other element and this exchange process is prescribed by the exchange matrix.

If there are only finitely many generators, the cluster algebra is of finite type. Finite type cluster algebras have been classified by Fomin and Zelevinsky ([13]). Their classification describes the finite type cluster algebras in terms of Dynkin diagrams.

More concretely: a seed is a pair $(\underline{x}, M)$ where $\underline{x} = \{x_1, \dots, x_n\}$ is a basis of $\mathbb{Q}(u_1, \dots, u_n)$ and $M = (M_{ij})_{ij}$ is a sign skew symmetric $n \times n$-matrix with integer entries, called the exchange matrix. That means that the sign of M_{ij} is the opposite of the sign of M_{ji}.

Then one defines an involutive map μ_k (for $k \in \{1, \dots, n\}$) on the seeds, called the mutation in direction of k, through $\mu_k(\underline{x}) = (x_1, \dots, \widehat{x}_k, x'_k, \dots, x_n)$ where x'_k is given by the relation

$$x_k \cdot x'_k = \prod_{\substack{x_i \in \underline{x} \\ M_{ik} > 0}} x_i^{M_{ik}} + \prod_{\substack{x_i \in \underline{x} \\ M_{ik} < 0}} x_i^{-M_{ik}}$$

In a similar way, one defines M' by

$$M'_{ij} := \begin{cases} -M_{ij} & \text{if } i = k \text{ or } j = k \\ M_{ij} + \frac{1}{2}(|M_{ik}|M_{kj} + M_{ik}|M_{kj}|) & \text{otherwise.} \end{cases}$$

and thus obtains $(\underline{x}', M')$ as $\mu_k((\underline{x}, M))$ (the matrix M' is also a sign skew symmetric $n \times n$-matrix over $\mathbb{Z}$). For more details we refer to Section 1 of the survey article [4] of Buan-Marsh. The x_i obtained through successive mutations are the so-called cluster variables. The cluster algebra $\mathcal{A} = \mathcal{A}(\underline{x}, M)$ is then defined as the algebra generated by the cluster variables. There can be infinitely many of them. Fomin-Zelevinsky have shown in [14] that $\mathcal{A}$ lies in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ (Laurent-phenomenon). First examples of cluster algebras are coordinate rings of SL_2, SL_3.

The field of cluster algebras is a young and very dynamic field. Since its first introduction, there have been many different directions in its development. We only mention a few connections to other areas (in parentheses: the objects corresponding to the cluster variables): the theory of Teichmüller spaces (Penner coordinates), see work of Fock-Goncharov, [9] and [10]; the representation theory of finite dimensional algebras (tilting modules), cf. [5]), triangulations of surfaces (diagonals), see the work [11] of Fomin-Shapiro-Thurston.

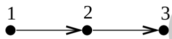
1.2. Cluster categories. — Cluster categories were introduced independently in the work [5] of Buan-Marsh-Reineke-Reiten-Todorov, and by Caldero-Chapoton-Schiffler, [6] to provide a categorification of the theory of cluster algebras.

We will use the approach of [5] to describe cluster categories and will consider the approach of [6]Êlater, cf. Section 2.

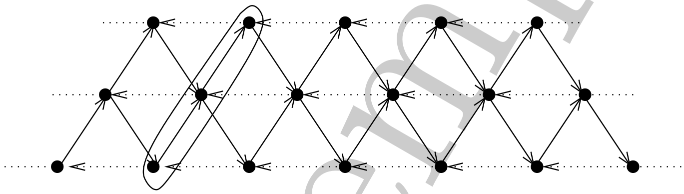
Let Q be a simply-laced Dynkin quiver (i.e. a quiver whose underlying graph is of type A, D or E). Let k be an algebraically closed field and kQ the path algebra of Q (for more details, we refer to the lecture notes of M. Brion, [3] in the same volume). Take the bounded derived category $\mathcal{D}^b(kQ)$ of finitely generated kQ-modules (for details on $\mathcal{D}^b(kQ)$ we refer to [15]) with shift denoted by [1] and Auslander-Reiten translate denoted by τ. By Happel ([15]), the category $\mathcal{D}^b(kQ)$ is triangulated, Krull-Schmidt, and has almost split sequences. To understand the category $\mathcal{D}^b(kQ)$ it is helpful to study its Auslander-Reiten quiver: The Auslander-Reiten quiver of a category is a combinatorial tool which helps understanding the category. Its vertices are by definition the indecomposable modules up to isomorphism and the number of arrows between two points are given by the dimension of the space of irreducible maps between two representatives of the corresponding modules.

We now associate a quiver $\mathbb{Z}Q$ to Q. Its vertices are (n, i) for $n \in \mathbb{Z}$, and where i a vertex of Q. For every arrow $i \to j$ in Q there are arrows $(n, i) \to (n, j)$ and $(n, j) \to (n+1, i)$ in $\mathbb{Z}Q$. So $\mathbb{Z}Q$ has the shape of a $\mathbb{Z}$-strip of copies of Q. Together with the map $\tau : (n, i) \to (n-1, i)$ ($n \in \mathbb{Z}$, i a vertex of Q), $\mathbb{Z}Q$ is a stable translation quiver as defined by Riedtmann (see [19]). For a precise definition of stable translation quivers we refer the reader to Section 3 below. We illustrate $\mathbb{Z}Q$ in Example 1.2. Happel has shown in [15], that the Auslander-Reiten quiver $\mathrm{AR}(\mathcal{D}^b(kQ))$ of $\mathcal{D}^b(kQ)$ is just $\mathbb{Z}Q$. In particular, the category $\mathcal{D}^b(kQ)$ is independent of the orientation of Q.

Example 1.2. — Let Q be a quiver of type A_3,

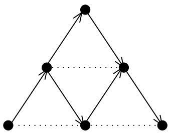
Q : 

Then, $\mathbb{Z}Q$ has the shape



with one copy of the quiver Q highlighted to show how it appears inside $\mathbb{Z}Q$. The dotted arrows indicate the Auslander-Reiten translate τ which sends each vertex to its leftmost neighbor. It is an auto-equivalence of the Auslander-Reiten quiver.

On the other hand, the Auslander-Reiten quiver of the module category kQ-mod of finitely generated kQ-modules looks like a triangle:

AR(kQ-mod): 

Observe that the infinite quiver $\mathbb{Z}Q$ can be viewed as being covered by copies of the Auslander-Reiten quiver of the module category kQ-mod, with additional arrows and dotted arrows introduced to connect the copies of the triangle of kQ-mod. With this picture in mind, we can describe the shift [1] on $\mathrm{AR}(\mathcal{D}^b(kQ))$: it sends each vertex to the "same" vertex in the next copy of the triangle AR(kQ-mod) to the right.

Back to the general situation, where Q is of simply-laced Dynkin type. The shift [1] then is the auto-equivalence of $\mathrm{AR}(\mathcal{D}^b(kQ))$ which sends a vertex to the corresponding vertex in the next copy of the Auslander-Reiten quiver of the module category kQ-mod and the translation τ sends a vertex to its leftmost neighbor. As an abbreviation, we write F for the auto-equivalence $\tau^{-1} \circ [1]$ of $\mathcal{D}^b(kQ)$. Now we are ready to define the cluster category associated to Q.