Séminaires & Congrès

COLLECTION SMF

ON SIMPLICITY AND STABILITY OF THE TANGENT BUNDLE OF RATIONAL HOMOGENEOUS VARIETIES

Ada Boralevi

GEOMETRIC METHODS IN REPRESENTATION THEORY, II

Numéro 25

Michel Brion, ed.

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
25, 2010, p. 273–295

ON SIMPLICITY AND STABILITY OF THE TANGENT BUNDLE OF RATIONAL HOMOGENEOUS VARIETIES

by

Ada Boralevi

Abstract. — Given a rational homogeneous variety G/P where G is complex simple and of type ADE, we prove that the tangent bundle $\mathrm{T}_{G/P}$ is simple, meaning that its only endomorphisms are scalar multiples of the identity. This result combined with Hitchin-Kobayashi correspondence implies stability of the tangent bundle with respect to the anticanonical polarization. Our main tool is the equivalence of categories between homogeneous vector bundles on G/P and finite dimensional representations of a given quiver with relations.

Résumé **(Sur la simplicité et la stabilité du fibré tangent des variétés rationelles homogènes)**
Soit G/P une variété homogène rationnelle, où G est un groupe de Lie simple, complexe et de type ADE. On démontre que le fibré tangent $T_{G/P}$ est simple, c'est-à-dire, ses seuls endomorphismes sont les multiples scalaires de l'identité. Notre théorème, combiné avec la correspondance de Hitchin-Kobayashi, implique la stabilité du fibré tangent par rapport à la polarisation anticanonique. L'instrument principal qu'on utilise est l'équivalence des catégories des fibrés vectoriels homogènes sur G/P et des représentations de dimension finie d'un carquois avec relations introduit par Bondal et Kapranov in 1990.

1. Introduction

In [18] Ramanan proved that irreducible homogeneous bundles on rational homogeneous varieties are stable, and hence in particular simple. If the underlying variety is Hermitian symmetric then this result applies to tangent bundles. For the general case, the Hitchin-Kobayashi correspondence gives a weaker result for the tangent bundle, polystability. In this paper we show that in fact the tangent bundle of any G/P is simple, where G is complex, simple and of type ADE. Simplicity and polystability combined give stability.
Our main tool is the equivalence of categories between homogeneous bundles on G/P

2000 Mathematics Subject Classification. — 14F05,14M17,14D20,16G20.
Key words and phrases. — Homogeneous vector bundle, simplicity, stability, quiver representation.

and finite dimensional representations of a given quiver with relations. Once the machinery of this equivalence of categories is set up, the simplicity of the tangent bundle turns out to be an immediate and surprisingly easy consequence of it. Indeed one only needs to look at endomorphisms of the bundle as endomorphisms of the associated quiver representation.

Homogeneous vector bundles have been classically studied using another equivalence of categories, namely that between homogeneous bundles on G/P and finite dimensional representations of the parabolic subgroup P.

In [3] Bondal and Kapranov had the idea of associating to any rational homogeneous variety a quiver with relations. By putting the appropriate relations one gets the aforementioned equivalence of categories between G-homogeneous vector bundles on G/P and finite dimensional representations of the quiver. The relations were later refined by Hille in [9].

In [1] Álvarez-Cónsul and García-Prada gave an equivalent construction, while in [17] Ottaviani and Rubei used the quiver for computing cohomology, obtaining a generalization of the well-known Borel-Weil-Bott theorem holding on Hermitian symmetric varieties of ADE type.

We describe both equivalences of categories and give details on the quiver, its relations and its representations in Sections 2 and 3.

Sections 4 and 5 contain results on simplicity and stability. We use the quiver to prove that homogeneous vector bundles whose associated quiver representation has a particular configuration—we call such bundles multiplicity free—are weakly simple, which means that their only G-invariant endomorphisms are scalar multiples of the identity. Our result holds on any G/P, where G is complex, simple and of type ADE:

Proposition A. — *Let E be a multiplicity free homogeneous vector bundle on G/P. Let k be the number of connected components of the quiver $\mathcal{Q}|_E$. Then $\mathrm{H}^0(\operatorname{End} E)^G = \mathbb{C}^k$. In particular if $\mathcal{Q}|_E$ is connected, then E is weakly simple.*

It turns out that the tangent bundle $\mathrm{T}_{G/P}$ of a rational homogeneous variety of ADE type is multiplicity free and connected, and that moreover the isotypical component $\mathrm{H}^0(\operatorname{End} \mathrm{T}_{G/P})^G$ coincides with the whole space $\mathrm{H}^0(\operatorname{End} \mathrm{T}_{G/P})$, or in other words that the bundle is simple.

Theorem B. — *Let $\mathrm{T}_{G/P}$ the tangent bundle on a rational homogeneous variety G/P, where G is a complex simple Lie group of type ADE and P one of its parabolic subgroups. Then $\mathrm{T}_{G/P}$ is simple.*

If algebraic geometry, representation theory and quiver representations give us simplicity, for stability differential geometry also joins the team. A homogeneous variety G/P is in fact also a Kähler manifold, and as such it admits a Hermite-Einstein structure. In virtue of the Hitchin-Kobayashi correspondence this is equivalent to the polystability of its tangent bundle. This together with Theorem B gives:

Theorem C. — *Let $\mathrm{T}_{G/P}$ be the tangent bundle on a rational homogeneous variety G/P, where G is a complex simple Lie group of type ADE, and P one of its parabolic* *subgroups. Then* $\mathrm{T}_{G/P}$ *is stable with respect to the anticanonical polarization* $-K_{G/P}$ *induced by the Hermite-Einstein structure.*

In the case where G/P is a point-hyperplane incidence variety in $\mathbb{P}^n$, we obtain a complete understanding of the stability of the tangent bundle with respect to different polarizations:

Proposition D. — *Let* $\mathbb{F} = \mathbb{F}(0, n-1, n)$ *be the point-hyperplane incidence variety, and set:*

$$m(n) = \frac{-n + n\sqrt{4n^2 + 4n - 3}}{2(n^2 + n - 1)}.$$

Then the tangent bundle $\mathrm{T}_{\mathbb{F}}$ *is stable with respect to the polarization* $\mathcal{O}_{\mathbb{F}}(a, b)$ *if and only if it is semistable if and only if* $m(n)a \leq b \leq m(n)^{-1}a$.

We also show similar computations for SL_4/B.

In the last Section 6 we deal with moduli spaces. We quote and generalize the results from [17], where the authors showed that King's notion of semistability [13] for a representation $[E]$ of the quiver $\mathcal{Q}_{G/P}$ is in fact equivalent to the Mumford-Takemoto semistability of the associated bundle E on G/P, when the latter is a Hermitian symmetric variety.

We can thus construct moduli spaces of G-homogeneous semistable bundles E with fixed $\operatorname{gr} E$ on any homogenous variety G/P of ADE type.

Acknowledgements. — This paper is part of my PhD thesis. I am very grateful to my advisor Giorgio Ottaviani for the patience with which he followed this work very closely and for always transmitting me lots of encouragement and mathematical enthusiasm. I would also like to thank Jean-Pierre Demailly for inviting me to Grenoble and for his warm hospitality and Michel Brion and the referee for many helpful comments.

2. Preliminaries

2.1. Notations and first fundamental equivalence of categories. — Let G be a complex semisimple Lie group. We make a choice $\Delta = \{\alpha_1, \dots, \alpha_n\}$ of simple roots of $\mathfrak{g} = \operatorname{Lie} G$ and thus of some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We call Φ^+ (respectively Φ^-) the set of positive (negative) roots. Then $\mathfrak{g}$ decomposes as:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \,.$$

A parabolic subgroup $P \leq G$ is a subgroup P conjugated to one of the standard parabolic subgroups $P(\Sigma)$, where:

$$\operatorname{Lie}(P(\Sigma)) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^-(\Sigma)} \mathfrak{g}_\alpha,$$

for a subset $\Sigma \subset \Delta$ that induces $\Phi^-(\Sigma) = \{\alpha \in \Phi^- | \alpha = \sum_{\alpha_i \notin \Sigma} p_i \alpha_i\}$.

If $\Sigma = \Delta$, then $P(\Delta) = B$ is the Borel subgroup.

A rational homogeneous variety is a quotient G/P.

A vector bundle E on G/P is called (G)-homogeneous if there is an action of G on E such that the following diagram commutes:

$$\begin{tikzcd} G \times E \arrow[r] \arrow[d] & E \arrow[d] \\ G \times G/P \arrow[r] & G/P \end{tikzcd}$$

where the bottom row is just the natural action of G on the cosets G/P.

Note that the tangent bundle $\mathrm{T}_{G/P}$ on any rational homogeneous variety G/P is obviously a G-homogeneous bundle.

The category of G-homogeneous vector bundles on G/P is equivalent to the category P-mod of representations of P, and also to the category of integral $\mathfrak{p}$-modules, where $\mathfrak{p} = \mathrm{Lie}(P)$, see for example [3].

More in detail, the group G is a principal bundle over G/P with fiber P. Any G-homogeneous vector bundle E of rank r is induced by this principal bundle via a representation $\rho : P \to \mathrm{GL}(r)$. We denote $E = E_\rho$. Indeed, E of rank r over G/P is homogeneous if and only if there exists a representation $\rho : P \to \mathrm{GL}(r)$ such that $E \simeq E_\rho$, and this entails the aforementioned equivalence of categories.

For any weight λ we denote by E_λ the homogeneous bundle corresponding to Σ_λ^*, the dual of the irreducible representation Σ_λ of the parabolic sugroup with maximal weight λ. Here λ belongs to the fundamental Weyl chamber of the reductive part of P. Indeed, P decomposes as $P = R \cdot N$ into a reductive part R and a unipotent part N. At the level of Lie algebras this decomposition entails a splitting $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{n}$, with the obvious notation $\mathfrak{r} = \mathrm{Lie}\, R$ and $\mathfrak{n} = \mathrm{Lie}\, N$. Moreover from a result by Ise [12] we learn that a representation of $\mathfrak{p}$ is completely reducible if and only if it is trivial on $\mathfrak{n}$, hence it is completely determined by its restriction to $\mathfrak{r}$.

The well-known Borel-Weil-Bott theorem [4] computes the cohomology of such E_λ's by using purely Lie algebra tools, namely the orbit of the weight λ under the action of the Weyl group. In particular the theorem states that if λ is G-dominant then $\mathrm{H}^0(E_\lambda)$ is the dual of the irreducible representation of G with highest weight λ and all higher cohomology vanishes.

3. The quiver $\mathcal{Q}_{G/P}$

3.1. Definition of the quiver $\mathcal{Q}_{G/P}$ and its representations. — Other than looking at homogeneous bundles as P-modules, it is useful to try a different point of view and look at these same bundles as representations of a given quiver with relations. For basics on quiver theory we refer the reader to [5] or [13].

To any rational homogeneous variety G/P we can associate a quiver with relations, that we denote by $\mathcal{Q}_{G/P}$. The idea is to exploit all the information given by the choice of the parabolic subgroup P, with its Levi decomposition $P = R \cdot N$.

Let Λ be the fundamental Weyl chamber of G, and let Λ^+ be the Weyl chamber of the reductive part R. Then we can give the following: