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PERVERSE SHEAVES AND MODULAR REPRESENTATION THEORY

by

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Abstract. — This paper is an introduction to the use of perverse sheaves with positive characteristic coefficients in modular representation theory. In the first part, we survey results relating singularities in finite and affine Schubert varieties and nilpotent cones to modular representations of reductive groups and their Weyl groups. The second part is a brief introduction to the theory of perverse sheaves with an emphasis on the case of positive characteristic and integral coefficients. In the final part, we provide some explicit examples of stalks of intersection cohomology complexes with integral or positive characteristic coefficients in nilpotent cones, mostly in type A. Some of these computations might be new.

***Résumé* (Faisceaux pervers et théorie des représentations modulaires).** — Cet article est une introduction à l'emploi des faisceaux pervers à coefficients en caractéristique non nulle en théorie des représentations modulaires. Dans la première partie, nous rappelons des résultats reliant les singularités des variétés de Schubert finies et affines, ainsi que celles des cônes nilpotents, aux représentations modulaires des groupes réductifs et de leurs groupes de Weyl. La deuxième partie est une brève introduction à la théorie des faisceaux pervers, l'accent étant mis sur le cas où l'anneau de coefficients est un corps de caractéristique non nulle, ou bien l'anneau des entiers. Dans la dernière partie, nous donnons des exemples explicites de calculs de fibres de complexes d'intersection à coefficients entiers ou en caractéristique non nulle dans des cônes nilpotents, essentiellement en type A. Certains de ces calculs sont peut-être nouveaux.

Introduction

Representation theory has a very different flavour in positive characteristic. When one studies the category of representations of a finite group or a reductive group over a field of characteristic 0 (e.g. $\mathbb{C}$), one of the first observations to be made is that this category is semi-simple, meaning that every representation is isomorphic to a direct sum of irreducible representations. This fundamental fact helps answer

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many basic questions, e.g. the dimensions of simple modules, character formulae, and tensor product multiplicities. However, when one considers representations over fields of positive characteristic (often referred to as "modular" representations) the resulting categories are generally not semi-simple. This makes their study considerably more complicated and in many cases even basic questions remain unanswered. [1]

It turns out that some questions in representation theory have geometric counterparts. The connection is obtained via the category of perverse sheaves, a certain category that may be associated to an algebraic variety and whose structure reflects the geometry of the underlying variety and its subvarieties. The category of perverse sheaves depends on a choice of coefficient field and, as in representation theory, different choices of coefficient field can yield very different categories.

Since the introduction of perverse sheaves it has been realised that many phenomena in Lie theory can be explained in terms of categories of perverse sheaves and their simple objects — intersection cohomology complexes. In studying representations of reductive groups and related objects, singular varieties arise naturally (Schubert varieties and their generalizations, nilpotent varieties, quiver varieties...). It turns out that the invariants of these singularities often carry representation theoretic information. For an impressive list of such applications, see [**42**]. This includes constructing representations, computing their characters, and constructing nice bases for them.

However, most of these applications use a field k of characteristic zero for coefficients. In this paper, we want to give the reader a flavour for perverse sheaves and intersection cohomology with positive characteristic coefficients.

In the first section of this article we survey three connections between modular representation theory and perverse sheaves.

The geometric setting for the first result — known as the geometric Satake theorem — is a space (in fact an "ind-scheme") associated to a complex reductive group G. This space, $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$, commonly referred to as the affine Grassmannian, is a homogeneous space for the algebraic loop group $G(\mathbb{C}((t)))$. Under the action of $G(\mathbb{C}[[t]])$, it breaks up as a union of infinitely many finite-dimensional orbits. Theorems of Lusztig [**35**], Ginzburg [**21**], Beilinson-Drinfeld [**6**], and Mirković-Vilonen [**43**] explain that encoded in the geometry of the affine Grassmannian and its orbit closures is the algebraic representation theory over any field (and even over the integers) of the split form of the reductive group $G^\vee$ with root data dual to that of G, also known as the Langlands dual group.

The second family of results that we discuss involves the geometry of the finite flag variety G/B where again G is a complex reductive group, and a generalization of it closely related to the affine Grassmannian known as the affine flag variety $G(\mathbb{C}((t)))/\mathcal{I}$. We describe theorems of Soergel [**50**] and Fiebig [**17**, **18**, **19**, **20**] which show that the geometry of these spaces can be used to understand the modular representation theory of the Langlands dual group $G_k^\vee$ for k a field of characteristic larger than the

[1] For an introduction to the modular representation theory of finite groups we recommend the third part of [**46**], and for that of reductive groups, [**24**].

Coxeter number of $G_k^\vee$. In doing so, Fiebig is able to give a new proof of the celebrated Lusztig conjecture with an explicit bound on the characteristic.

The third theorem to be discussed is centered around the geometry of the variety $\mathcal{N}$ of nilpotent elements of a Lie algebra $\mathfrak{g}$, known as the nilpotent cone. The nilpotent cone has a natural resolution and, in 1976, Springer [51] showed that the Weyl group acts on the ℓ-adic cohomology of the fibers of this resolution. He showed moreover that from this collection of representations one could recover all of the irreducible ℓ-adic representations and that they came with a natural labelling by a nilpotent adjoint orbit with an irreducible G-equivariant local system. This groundbreaking discovery was followed by a series of related constructions, one of which, based on the Fourier-Deligne transform, has recently been used by the first author [25] to establish a modular version of the Springer correspondence.

The second goal of this article which occupies the second and third sections is to provide an introduction to "modular" perverse sheaves, in other words perverse sheaves with coefficients in a field of positive characteristic. We begin by recalling the theory of perverse sheaves, highlighting the differences between characteristic zero and characteristic p, and also the case of integer coefficients. We treat in detail the case of the nilpotent cone of $\mathfrak{sl}_2$.

In the last part, we treat more examples in nilpotent cones. We calculate all the IC stalks in all characteristics $\neq 3$ for the nilpotent cone of $\mathfrak{sl}_3$, and all the IC stalks in all characteristics $\neq 2$ for the subvariety of the nilpotent cone of $\mathfrak{sl}_4$ consisting of the matrices which square to zero. Before that, we recall how to deal with simple and minimal singularities in type A, for two reasons: we need them for the three-strata calculations, and they can be dealt with more easily than for arbitrary type (which was done in [26, 27]). As a complement, we give a similar direct approach for a minimal singularity in the nilpotent cone of $\mathfrak{sp}_{2n}$.

The first two parts partly correspond to the talks given by the first and third author during the summer school, whose titles were "Intersection cohomology in positive characteristic I, II". The third part contains calculations that the three of us did together while in Grenoble. These calculations were the first non-trivial examples involving three strata that we were able to do.

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1. Motivation

Perverse sheaves with coefficients in positive characteristic appear in a number of different contexts as geometrically encoding certain parts of modular representation theory. This section will provide a survey of three examples of this phenomenon: the geometric Satake theorem, the work of Soergel and Fiebig on Lusztig's conjecture, and the modular Springer correspondence. The corresponding geometry for the three pictures will be respectively affine Grassmannians, finite and affine flag varieties, and nilpotent cones.

Throughout, $G_{\mathbb{Z}}$ will denote a split connected reductive group scheme over $\mathbb{Z}$. Given a commutative ring k, we denote by G_k the split reductive group scheme over k obtained by extension of scalars

$$G_k = \operatorname{Spec} k \times_{\operatorname{Spec} \mathbb{Z}} G_{\mathbb{Z}}.$$

In Sections 1.1 and 1.2 we will consider $G_{\mathbb{C}}$, while in Section 1.3, we will consider $G_{\mathbb{F}_q}$.

We fix a split maximal torus $T_{\mathbb{Z}}$ in $G_{\mathbb{Z}}$. We denote by

$$(X^*(T_{\mathbb{Z}}), R^*, X_*(T_{\mathbb{Z}}), R_*)$$

the root datum of $(G_{\mathbb{Z}}, T_{\mathbb{Z}})$. We denote by $(G^\vee_{\mathbb{Z}}, T^\vee_{\mathbb{Z}})$ the pair associated to the dual root datum. Thus $G^\vee_{\mathbb{Z}}$ is the Langlands dual group. In Subsections 1.1 and 1.2, we will consider representations of $G^\vee_k = \operatorname{Spec} k \times_{\operatorname{Spec} \mathbb{Z}} G^\vee_{\mathbb{Z}}$, where k can be, for example, a field of characteristic p. We have $X^*(T^\vee_{\mathbb{Z}}) = X_*(T_{\mathbb{Z}})$ and $X_*(T^\vee_{\mathbb{Z}}) = X^*(T_{\mathbb{Z}})$.

We also fix a Borel subgroup $B_{\mathbb{Z}}$ of $G_{\mathbb{Z}}$ containing $T_{\mathbb{Z}}$. This determines a Borel subgroup $B^\vee_{\mathbb{Z}}$ of $G^\vee_{\mathbb{Z}}$ containing $T^\vee_{\mathbb{Z}}$. This also determines bases of simple roots $\Delta \subset R^*$ and $\Delta^\vee \subset R_*$. It will be convenient to choose $\Delta_* := -\Delta^\vee$ as a basis for R_* instead, so that the coroots corresponding to $B^\vee_{\mathbb{Z}}$ are the negative coroots $R_*^- = -R_*^+$.

1.1. The geometric Satake theorem. — In this subsection and the next one, to simplify the notation, we will identify the group schemes $G_{\mathbb{C}} \supset B_{\mathbb{C}} \supset T_{\mathbb{C}}$ with their groups of $\mathbb{C}$-points $G \supset B \supset T$.

We denote by $\mathcal{K} = \mathbb{C}((t))$ the field of Laurent series and by $\mathcal{O} = \mathbb{C}[[t]]$ the ring of Taylor series. The affine (or loop) Grassmannian $\mathcal{G}r = \mathcal{G}r^G$ is the homogeneous space $G(\mathcal{K})/G(\mathcal{O})$. It has the structure of an ind-scheme. In what follows we will attempt to sketch a rough outline of this space and then briefly explain how perverse sheaves on it are related to the representation theory of $G^\vee_k$, where k is any commutative ring of finite global dimension. We refer the reader to [4, 6, 32, 43] for more details and proofs.

We have a natural embedding of the coweight lattice $X_*(T) = \operatorname{Hom}(\mathbb{G}_m, T)$ into the affine Grassmannian: each $\lambda \in X_*(T)$ defines a point t^λ of $G(\mathcal{K})$ via

$$\operatorname{Spec} \mathcal{K} = \operatorname{Spec} \mathbb{C}((t)) \xrightarrow{c} \mathbb{G}_m = \operatorname{Spec} \mathbb{C}[t, t^{-1}] \xrightarrow{\lambda} T \xrightarrow{i} G$$

where c comes from the inclusion $\mathbb{C}[t, t^{-1}] \hookrightarrow \mathbb{C}((t))$ and $i : T \to G$ is the natural inclusion, and hence a point $[t^\lambda]$ in $\mathcal{G}r = G(\mathcal{K})/G(\mathcal{O})$.

For example, when $G = GL_n$ and T is the subgroup of diagonal matrices the elements of $X_*(T)$ consist of n-tuples of integers $\lambda = (\lambda_1, \dots, \lambda_n)$ and they sit inside