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ON THE STRUCTURE OF THE CATEGORY $\mathcal{O}$ FOR $\mathcal{W}$-ALGEBRAS

by

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Abstract. — A W-algebra (of finite type) $\mathcal{W}$ is a certain associative algebra associated with a semisimple Lie algebra, say $\mathfrak{g}$, and its nilpotent element, say e. The goal of this paper is to study the category $\mathcal{O}$ for $\mathcal{W}$ introduced by Brundan, Goodwin and Kleshchev. We establish an equivalence of this category with a certain category of $\mathfrak{g}$-modules. In the case when e is of principal Levi type (this is always so when $\mathfrak{g}$ is of type A) the category of $\mathfrak{g}$-modules in interest is the category of generalized Whittaker modules introduced by McDowell, and studied by Milicic-Soergel and Backelin.

***Résumé* (Sur la structure de la catégorie $\mathcal{O}$ de $\mathcal{W}$-algèbres).** — Une W-algèbre (de type fini) $\mathcal{W}$ est une certaine algèbre associative associée à une algèbre de Lie semisimple $\mathfrak{g}$ et à un élément nilpotent $e \in \mathfrak{g}$. Le but de l'article est d'étudier la catégorie $\mathcal{O}$ de $\mathcal{W}$, introduite par Brundan, Goodwin et Kleshchev. Nous obtenons une équivalence de cette catégorie avec une certaine catégorie de $\mathfrak{g}$-modules. Lorsque e est principal dans une sous-algèbre de Levi de $\mathfrak{g}$ (ce qui est toujours le cas en type A), cette catégorie de $\mathfrak{g}$-modules est la catégorie des modules de Whittaker généralisés introduite par McDowell, et étudiée par Milicic-Soergel et Backelin.

1. Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field $\mathbb{K}$ of characteristic zero. Choose a nilpotent element $e \in \mathfrak{g}$. Associated to the pair $(\mathfrak{g}, e)$ is a certain associative algebra $\mathcal{W}$, which is closely related to the universal enveloping algebra $U(\mathfrak{g})$. It was studied extensively during the last decade starting from Premet's paper [19], see also [5, 6, 7, 9, 10, 13, 15], [20]-[21]. Definitions of a W-algebra due to Premet, [19], and the author, [15], are recalled in Section 2.

In the representation theory of $U(\mathfrak{g})$ a crucial role is played by the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of $U(\mathfrak{g})$-modules. In particular, all finite dimensional

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$U(\mathfrak{g})$-modules and all Verma modules belong to $\mathcal{O}$. In [5] Brundan, Goodwin and Kleshchev introduced the notion of the category $\mathcal{O}$ for $\mathcal{W}$. This category also contains all finite dimensional $\mathcal{W}$-modules as well as analogs of Verma modules. See Section 3 for definitions.

The BGK category $\mathcal{O}$ is not always very useful. For example, for a *distinguished* nilpotent element $e \in \mathfrak{g}$ (i.e., such that the centralizer $\mathfrak{z}_{\mathfrak{g}}(e)$ contains no nonzero semisimple elements) $\mathcal{O}$ consists precisely of finite dimensional modules. The other extreme is the case when e is of principal Levi type. This means that there is a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ such that e is a principal nilpotent element in $\mathfrak{l}$. Here the BGK category $\mathcal{O}$ looks quite similar to the BGG one.

In [5], Conjecture 5.3, the authors conjectured that for e of principal Levi type there exists a category equivalence between their category $\mathcal{O}$ and a certain category of *generalized Whittaker modules* introduced by McDowell, [16], and studied by Milicic and Soergel, [17], and Backelin, [1]. We postpone the description of this category until Section 4. The main result of this paper, Theorem 4.1, gives a proof of that conjecture.

Let us describe the content of this paper. In Section 2 we recall the definition of W-algebras and the basic theorem of our paper [15], the so called decomposition theorem. In Section 3 the notion of the category $\mathcal{O}$ for a W-algebra is recalled. In Section 4 we introduce the category of generalized Whittaker modules. Special cases of this category are, firstly, Skryabin's category of Whittaker modules (or, more precisely, the full subcategory there consisting of all finitely generated modules) and, secondly, the categories studied in [1, 16, 17]. Then we state the category equivalence theorem 4.1 generalizing the Skryabin equivalence theorem from the appendix to [19] and proving the conjecture of Brundan, Goodwin and Kleshchev. The proof of Theorem 4.1 is given in Section 5. Essentially, we generalize the proof of the Skryabin equivalence theorem given in [15], Subsection 3.3, checking that certain topological algebras are isomorphic.

Finally, in Section 6 we will describe some applications of our results.

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2. W-algebras

Throughout the paper everything is defined over an algebraically closed field $\mathbb{K}$ of characteristic 0.

Let G be a reductive algebraic group, $\mathfrak{g}$ its Lie algebra, and $\mathcal{U}$ the universal enveloping algebra of $\mathfrak{g}$. Fix a nonzero nilpotent element $e \in \mathfrak{g}$. Choose an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$ and set $Q := Z_G(e, h, f)$. Denote by T a maximal torus of Q. Also introduce a grading on $\mathfrak{g}$ by eigenvalues of $\operatorname{ad} h$: $\mathfrak{g} := \bigoplus \mathfrak{g}(i), \mathfrak{g}(i) := \{\xi \in \mathfrak{g} | [h,\xi] = i\xi\}$. Consider the one-parameter subgroup $\gamma : \mathbb{K}^\times \to G$ with $\frac{d}{dt}|_{t=0}\gamma = h$. Choose a G-invariant symmetric form $(\cdot,\cdot)$ on $\mathfrak{g}$, whose restriction to any algebraic reductive subalgebra is non-degenerate. This form allows one to identify $\mathfrak{g} \cong \mathfrak{g}^*$. Let $\chi = (e,\cdot)$ be the element of $\mathfrak{g}^*$ corresponding to e.

Equip the space $\mathfrak{g}(-1)$ with a symplectic form ω_χ as follows: $\omega_\chi(\xi,\eta) = \langle\chi,[\xi,\eta]\rangle$. Fix a lagrangian subspace $l \subset \mathfrak{g}(-1)$ and define the subalgebra $\mathfrak{m} := l \oplus \bigoplus_{i\leqslant -2}\mathfrak{g}(i) \subset \mathfrak{g}$. According to Premet, [19], a W-algebra $\mathcal{W}$ associated with e is, by definition, $(\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\operatorname{ad}\mathfrak{m}}$, where $\mathfrak{m}_\chi := \{\xi - \langle\chi,\xi\rangle, \xi\in\mathfrak{m}\}$. As Gan and Ginzburg checked in [9], $\mathcal{W}$ does not depend on the choice of l up to some natural isomorphism. Thus we can choose a T-stable lagrangian subspace $l \subset \mathfrak{g}(-1)$ so we get an action of T on $\mathcal{W}$. Note that the image of $\mathfrak{t}$ in $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi$ consists of $\operatorname{ad}\mathfrak{m}$-invariants, for $\mathfrak{m}$ is $\mathfrak{t}$-stable and χ is annihilated by $\mathfrak{t}$. So we get an embedding $\mathfrak{t} \hookrightarrow \mathcal{W}$. It is compatible with the action of T in the sense that the differential of the T-action coincides with the adjoint action of $\mathfrak{t} \subset \mathcal{W}$. In fact, from the construction in [9] it follows that Q acts on $\mathcal{W}$ by algebra automorphisms, see [20], Subsection 2.2, for details.

One important feature of $\mathcal{W}$ is that the category $\mathcal{W}$-Mod of (left) $\mathcal{W}$-modules is equivalent to a certain full subcategory in $\mathcal{U}$-Mod to be described now. We say that a left $\mathcal{U}$-module M is a *Whittaker* module if $\mathfrak{m}_\chi$ acts on M by locally nilpotent endomorphisms. In this case $M^{\mathfrak{m}_\chi} = \{m \in M | \xi m = \langle\chi,\xi\rangle m, \forall \xi\in\mathfrak{m}\}$ is a $\mathcal{W}$-module. As Skryabin proved in the appendix to [19], the functor $M \mapsto M^{\mathfrak{m}_\chi}$ is an equivalence between the category of Whittaker $\mathcal{U}$-modules and $\mathcal{W}$-Mod. A quasiinverse equivalence is given by $N \mapsto \mathcal{S}(N) := (\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)\otimes_{\mathcal{W}} N$, where $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi$ is equipped with a natural structure of a $\mathcal{U}$-$\mathcal{W}$-bimodule.

Note also that the center of $\mathcal{W}$ can be identified with the center $\mathcal{Z}$ of $\mathcal{U}$, as follows. It is clear that $\mathcal{Z} \subset \mathcal{U}^{\operatorname{ad}\mathfrak{m}}$ whence we have a homomorphism $\mathcal{Z} \to \mathcal{W}$. This homomorphism is injective and its image coincides with the center of $\mathcal{W}$, see [20], the footnote to the Question 5.1.

An alternative description of $\mathcal{W}$ was given in [15]. Define the Slodowy slice $S := e + \mathfrak{z}_\mathfrak{g}(f)$. It will be convenient for us to consider S as a subvariety in $\mathfrak{g}^*$. Define the *Kazhdan* action of $\mathbb{K}^\times$ on $\mathfrak{g}^*$ by $t.\alpha = t^{-2}\gamma(t)\alpha$. This action preserves S and, moreover, $\lim_{t\to\infty} t.s = \chi$ for all $s \in S$. Also note that Q acts on S in a natural way.

Set $V := [\mathfrak{g}, f]$. Equip V with the symplectic form $\omega(\xi,\eta) = \langle\chi,[\xi,\eta]\rangle$, the action of $\mathbb{K}^\times, t.v = \gamma(t)^{-1}v$, and the natural action of Q.

Now let Y be a smooth affine variety equipped with commuting actions of a reductive group Q and of the one-dimensional torus $\mathbb{K}^\times$. For instance, one can take $Y = \mathfrak{g}^*, S, V^*$ equipped with the natural actions of $Q = Z_G(e,h,f)$ and the Kazhdan actions of $\mathbb{K}^\times$. Note that the grading on $\mathbb{K}[S]$ induced by the Kazhdan action is positive.

As follows from the explanation in [13], Subsection 2.1, for $Y = \mathfrak{g}^*, V^*, S$ there are certain *star-products* $* : \mathbb{K}[Y]\otimes\mathbb{K}[Y] \to \mathbb{K}[Y][\hbar], f * g = \sum_{i=0}^\infty D_i(f,g)\hbar^{2i}$, satisfying the following conditions.

1. $*$ is associative, that is, a natural extension of $*$ to $\mathbb{K}[Y][\hbar]$ turns $\mathbb{K}[Y][\hbar]$ into an associative $\mathbb{K}[\hbar]$-algebra, and 1 is a unit for this product.
2. $D_0(f,g) = fg$ for all $f,g \in \mathbb{K}[Y]$.
3. $D_i(\cdot,\cdot)$ is a bidifferential operator of order at most i in each variable.
4. $*$ is a Q-equivariant map $\mathbb{K}[Y]\otimes\mathbb{K}[Y] \to \mathbb{K}[Y][\hbar]$.
5. $*$ is homogeneous with respect to $\mathbb{K}^\times$. This, by definition, means that the degree of D_i is $-2i$ for all i.
6. There is a Q-equivariant map $\mathfrak{q} \to \mathbb{K}[Y][\hbar], \xi \mapsto \widehat{H}_\xi$, such that $\hbar^{-2}[\widehat{H}_\xi, \bullet]$ coincides with the image of ξ under the differential of the Q-action on $\mathbb{K}[Y][\hbar]$.

This construction allows one to equip $\mathbb{K}[\mathfrak{g}^*], \mathbb{K}[V^*], \mathbb{K}[S]$ with new associative products $*_1$ defined by $f *_1 g = \sum_{i=0}^\infty D_i(f,g)$. The algebras $\mathbb{K}[\mathfrak{g}^*], \mathbb{K}[V^*], \mathbb{K}[S]$ with these new products are T (and, in fact, Q)-equivariantly isomorphic to $\mathcal{U}$, the Weyl algebra $\mathbf{A}_V$ of the vector space V, and the W-algebra $\mathcal{W}$, respectively.

We finish this section by recalling a decomposition result from [15], which plays a crucial role in our construction.

Recall that if X is an affine algebraic variety and x a point of X we can consider the completion $\mathbb{K}[X]^\wedge_x := \varprojlim_k \mathbb{K}[X]/\mathbb{K}[X]\mathfrak{m}_x^k$, where $\mathfrak{m}_x$ denotes the maximal ideal corresponding to x. If X is an affine space, then taking x for the origin and choosing a basis in X, we can identify $\mathbb{K}[X]^\wedge_x$ with the algebra of formal power series. The algebra $\mathbb{K}[X]^\wedge_x$ is equipped with the topology of the inverse image. If $D : \mathbb{K}[X]\otimes\mathbb{K}[X] \to \mathbb{K}[X]$ is a bidifferential operator, then it can be uniquely extended to a continuous bidifferential operator $\mathbb{K}[X]^\wedge_x \otimes \mathbb{K}[X]^\wedge_x \to \mathbb{K}[X]^\wedge_x$.

Since our star-products satisfy (3), we can extend them to the completions $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi, \mathbb{K}[V^*]^\wedge_0, \mathbb{K}[S]^\wedge_\chi$. So we get new algebra structures on $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]], \mathbb{K}[V^*]^\wedge_0[[\hbar]], \mathbb{K}[S]^\wedge_\chi[[\hbar]]$. These algebras have unique maximal ideals, for instance, the maximal ideal $\widetilde{\mathfrak{m}} \subset \mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]]$ is the inverse image of the maximal ideal in $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi$. The algebra $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi$ is complete in the $\widetilde{\mathfrak{m}}$-adic topology. The similar claims hold for the other two algebras.

Consider the algebra $\mathbb{K}[S]^\wedge_\chi[[\hbar]] \otimes_{\mathbb{K}[[\hbar]]} \mathbb{K}[V^*]^\wedge_0[[\hbar]]$ and let $\widetilde{\mathfrak{m}}$ denote its maximal ideal corresponding to the point $(\chi,0)$. Note that the algebra is not complete in the $\widetilde{\mathfrak{m}}$-adic topology. Taking the completion, we get the *completed tensor product*, which we denote by $\mathbb{K}[S]^\wedge_\chi[[\hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathbb{K}[V^*]^\wedge_0[[\hbar]]$. As a vector space, the last algebra is just $\mathbb{K}[S\times V^*]^\wedge_{(\chi,0)}[[\hbar]]$.

Finally, note that there is a natural identification $\varphi : \mathfrak{z}_\mathfrak{g}(e)\oplus V \to \mathfrak{g}, (\xi,\eta)\mapsto \xi+\eta$.

The first two assertions of the following Proposition follow from [15], Theorem 3.1.3, and the third follows from [13], Theorem 2.3.1, for semisimple G and from Remark 2.3.2 for a general reductive group G.

Proposition 2.1. — *There is a $Q\times\mathbb{K}^\times$-equivariant isomorphism*

$$\Phi_\hbar : \mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]] \to \mathbb{K}[S]^\wedge_\chi[[\hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\mathbb{K}[V^*]^\wedge_0[[\hbar]]$$

of topological $\mathbb{K}[[\hbar]]$-algebras satisfying the following conditions:

1. *$\Phi_\hbar(\sum_{i=0}^\infty f_i\hbar^{2i})$ contains only even powers of $\hbar$.*