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EXAMPLES FOR EXCEPTIONAL SEQUENCES OF INVERTIBLE SHEAVES ON RATIONAL SURFACES

by

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Abstract. — In this article we survey recent results of joint work with Lutz Hille on exceptional sequences of invertible sheaves on rational surfaces. This survey is supplemented by explicit examples.

Résumé **(Exemples de séquences exceptionnelles de faisceaux inversibles sur les surfaces rationnelles)**

Cet article présente une vue d'ensemble de notre travail en collaboration avec Lutz Hille sur les suites exceptionelles de faisceaux inversibles sur les surfaces rationelles. Par ailleurs, nous présentons quelques exemples explicites supplémentaires.

1. Introduction

The purpose of this note is to give a survey on the results of joint work with Lutz Hille [11] and to provide some explicit examples. The general problem addressed in [11] is to understand the derived category of coherent sheaves on an algebraic variety (for an introduction and overview on derived categories over algebraic varieties we refer to [12]; see also [8]). An important approach to understand derived categories is to construct exceptional sequences:

Definition **([1]).** — A coherent sheaf $\mathcal{E}$ on X an algebraic variety X is called *exceptional* if $\mathcal{E}$ is simple and $\operatorname{Ext}^i_X(\mathcal{E}, \mathcal{E}) = 0$ for every $i \neq 0$. A sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ of exceptional sheaves is called an *exceptional* sequence if $\operatorname{Ext}^k_X(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all k and for all $i > j$. If an exceptional sequence generates $D^b(X)$, then it is called *full*. A *strongly* exceptional sequence is an exceptional sequence such that $\operatorname{Ext}^k_X(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all $k > 0$ and all i, j.

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In [11] we have considered exceptional and strongly exceptional sequences on rational surfaces which consist of invertible sheaves. Though this seems to be a quite restrictive setting, it still covers a large and interesting class of varieties, and the results uncover interesting aspects of the derived category which had not been noticed so far. In particular, it seems that toric geometry plays an unexpectedly important role.

Besides giving a survey, we also want to consider in these notes some aspects related to noncommutative geometry, which have not been touched in [11]. By results of Bondal [5], the existence of a full strongly exceptional sequence implies the existence of an equivalence of categories

$$\mathbf{R}\mathrm{Hom}(\mathcal{T}, \,.\,) : D^b(X) \longrightarrow D^b(A-\mathrm{mod}),$$

where $\mathcal{T} := \bigoplus_{i=1}^n \mathcal{E}_i$, which is sometimes called a tilting sheaf, and $A := \mathrm{End}(\mathcal{T})$. This way, a strongly exceptional sequence provides a non-commutative model for X. The algebra A is finite and can be described as a path algebra with relations. One of the themes of this note will be to provide some interesting and explicit examples of such algebras.

2. Exceptional sequences and toric systems

Our general setting will be that X is a smooth complete rational surface defined over some algebraically closed field. We will always denote n the rank of the Grothendieck group of X. Note that $\mathrm{Pic}(X)$ is a free $\mathbb{Z}$-module of rank $n-2$. We are interested in exceptional sequences of invertible sheaves. That is, we are looking for divisors $E_1, \ldots, E_n$ on X such that the sheaves $\mathcal{O}(E_1), \ldots, \mathcal{O}(E_n)$ form a (strongly) exceptional sequence. In this situation, computation of Ext-groups reduces to compute cohomologies of divisors, i.e. for any two invertible sheaves $\mathcal{L}$, $\mathcal{M}$, there exists an isomorphism

$$\mathrm{Ext}^k_X(\mathcal{L}, \mathcal{M}) \cong H^k(X, \mathcal{L}^* \otimes \mathcal{M})$$

for every k. So, the $\mathcal{O}(E_i)$ to form a (strongly) exceptional sequence, it is necessary in addition that $H^k\big(X, \mathcal{O}(E_j - E_i)\big) = 0$ for every k and every $i > j$ (or for every $k > 0$ and every i, j, respectively). For simplifying notation, we will usually omit references to X and for some divisor D on X we write $h^k(D)$ instead of $\dim H^k\big(X, \mathcal{O}(D)\big)$. Note that for any divisor D and any blow-up $b : X' \to X$ the vector spaces $H^k\big(X, \mathcal{O}(D)\big)$ and $H^k\big(X', b^*\mathcal{O}(D)\big)$ are naturally isomorphic. This way there will be no ambiguities on where we determine the cohomologies of D.

In essence, we are facing a quite peculiar problem of cohomology vanishing: constructing a strongly exceptional sequence of invertible sheaves is equivalent to finding a set of divisors $E_1, \ldots, E_n$ such that $h^k(E_i - E_j) = 0$ for all $k > 0$ and all i, j. Formulated like this, the problem looks somewhat ill-defined, as it seems to imply a huge numerical complexity (the Picard group has rank $n-2$ and one has to check cohomology vanishing for $\sim n^2$ divisors). This can be tackled rather explicitly for small examples (see [10]), but such a procedure becomes very unwieldy in general.

Simple examples show that standard arguments for cohomology vanishing via Kawamata-Viehweg type theorems usually are not sufficient to solve the problem (however, see [16] for another kind of vanishing theorem which may be helpful here). So, to deal more effectively with the problem, we have to make more use of the geometry of X. Recall that $\mathrm{Pic}(X)$ (or better, $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$) is endowed with a bilinear quadratic form of signature $(1, -1, \ldots, -1)$, which is given by the intersection form. Moreover, the geometry of X gives us a canonical quadratic form on $\mathrm{Pic}(X)$, the Euler form χ, which by the Riemann-Roch theorem is of the form

$$\chi(D) = 1 + \frac{1}{2}(D^2 - K_X.D)$$

for $D \in \mathrm{Pic}(X)$. The condition that $E_1, \ldots, E_n$ yield an exceptional sequence implies that $\chi(E_j - E_i) = 0$ for every $i > j$. So, we get another bit of information which tells us that we should look for divisors which are sitting on a quadratic hypersurface in $\mathrm{Pic}(X)$ which is given by the equation $\chi(-D) = 0$ for $D \in \mathrm{Pic}(X)$. However, looking for integral solutions of a quadratic equation still does not help very much. To improve the situation, we have to exploit the Euler form more effectively. Considering its symmetrization and antisymmetrization:

$$\chi(D) + \chi(-D) = 2 + D^2 \quad \text{and}$$
$$\chi(D) - \chi(-D) = -K_X.D,$$

we get immediately:

Lemma 2.1. — *Let $D, E \in \mathrm{Pic}(X)$ such that $\chi(-D) = \chi(-E) = 0$, then*

(i) $\chi(D) = -K_X.D = D^2 + 2$;

(ii) $\chi(-D - E) = 0$ *iff* $E.D = 1$ *iff* $\chi(D) + \chi(E) = \chi(D + E)$.

Now, using Lemma 2.1, we can bring an exceptional sequence $\mathcal{O}(E_1), \ldots, \mathcal{O}(E_n)$ into a convenient normal form. For this, we pass to the difference vectors and set $A_i := E_{i+1} - E_i$ for $1 \leq i < n$ and

$$A_n := -K_X - \sum_{i=1}^{n-1} A_i.$$

By Lemma 2.1, we get:

(i) $A_i.A_{i+1} = 1$ for $1 \leq i \leq n$;

(ii) $A_i.A_j = 0$ for $i \neq j$ and $\{i, j\} \neq \{k, k+1\}$ for some $1 \leq k \leq n$;

(iii) $\sum_{i=1}^{n} A_i = -K_X$.

Note that here the indices are to be read in the cyclic sense, i.e. we identify integers modulo n. This leads to the following definition:

Definition 2.2. — We call a set of divisors on X which satisfy conditions (i), (ii), and (iii) above a *toric system*.

A toric system seems to be the most efficient form to encode an exceptional sequence of invertible sheaves. But why the name? It turns out that a toric system encodes a smooth complete rational surface. Consider the following short exact sequence:

$$0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{A} \mathbb{Z}^n \longrightarrow \mathbb{Z}^2 \longrightarrow 0,$$

where A maps a divisor class D to the tuple $(A_1.D, \ldots, A_n.D)$. Denote $l_1, \ldots, l_n$ the images of the standard basis fo $\mathbb{Z}^n$ in $\mathbb{Z}^2$. It is shown in [11] that the l_i form a cyclically ordered set of primitive elements in $\mathbb{Z}^2$ such that $l_i, l_i + 1$ forms a basis for $\mathbb{Z}^2$ for every i. This is precisely the defining data for a smooth complete toric surface. On the other hand, the construction of the l_i from the A_i is an example for *Gale duality*. In particular, if we dualize above short exact sequence, we obtain a standard sequence from toric geometry:

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{L} \mathbb{Z}^n \longrightarrow \mathrm{Pic}(Y) \longrightarrow 0,$$

where Y denotes the toric surface associated to the toric system $A_1, \ldots, A_n$ and L the matrix whose rows are the l_i which act via the standard Euklidian inner product as linear forms on $\mathbb{Z}^2$. By this duality, $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$ and their intersection products can canonically be identified. So we get the following remarkable structural result:

Theorem 2.3 **([11])**. — *Let X be a smooth complete rational surface, let $\mathcal{O}_X(E_1), \ldots, \mathcal{O}_X(E_n)$ be a full exceptional sequence of invertible sheaves on X, and set $E_{n+1} := E_1 - K_X$. Then to this sequence there is associated in a canonical way a smooth complete toric surface with torus invariant prime divisors $D_1, \ldots, D_n$ such that $D_i^2 + 2 = \chi(E_{i+1} - E_i)$ for all $1 \leq i \leq n$.*

This theorem at once gives us information about the possible values for $\chi(A_i)$ and the combinatorial types of quivers associated to strongly exceptional sequences, but it does not give us a method for constructing them. This will be discussed in the next section. We conclude with some remarks on some ambiguities related to toric systems. Of course, for a given exceptional sequence $\mathcal{O}(E_1), \ldots, \mathcal{O}(E_n)$, the associated toric system $A_1, \ldots, A_n$ is unique. However, from $A_1, \ldots, A_n$ we will get back the original sequence only up to twist, i.e. by summing up the A_i we get $\mathcal{O}_X, \mathcal{O}(A_1), \mathcal{O}(A_1 + A_2), \ldots, \mathcal{O}(\sum_{i=1}^{n-1} A_i)$. Of course, this does not really matter, as we are interested only in the differences among the divisors.

By construction, the condition that the cohomologies vanish a priori apply only to the A_i with $1 \leq i < n$, but it follows from Serre duality that also $h^k(-A_n) = 0$ for all k. More generally, it follows that every cyclic enumeration of the A_i gives rise to an exceptional sequence. Moreover, if $A_1, \ldots, A_n$ yields an exceptional sequence, then so does $A_n, \ldots, A_1$.

The case where the E_i form a strongly exceptional sequence has less symmetries in general. In particular, we cannot expect that the higher cohomologies of A_n vanish