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Olivier Serman

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# ORTHOGONAL AND SYMPLECTIC BUNDLES ON CURVES AND QUIVER REPRESENTATIONS

*by*

Olivier Serman

***Abstract.*** — We show how quiver representations and their invariant theory naturally arise in the study of some moduli spaces parametrizing bundles defined on an algebraic curve, and how they lead to fine results regarding the geometry of these spaces.

***Résumé*** **(Fibrés orthogonaux et symplectiques sur les courbes et représentations de carquois)**
On montre comment la théorie des représentations de carquois apparaît naturellement lors de l'étude des espaces de modules de fibrés principaux définis sur une courbe algébrique, et comment elle permet d'analyser la géométrie de ces variétés.

## Introduction

Let $X$ be a smooth projective curve defined over an algebraically closed field $k$ of characteristic 0. It is a natural question to try to find an algebraic variety which parametrizes objects of some given kind defined on the curve $X$.

A first example is provided by the study of line bundles of degree 0 on $X$. It has been known essentially since Abel and Jacobi that there is actually an abelian variety, the Jacobian variety $J_X$, which parametrizes line bundles of degree 0 on $X$. We know a great deal about this variety, whose geometry is closely related to the geometry of $X$.

Weil's suggestion in [34] that vector bundles (which appear in his paper as "$\mathbf{GL}_r$-divisors") should provide a relevant non-abelian analogue of this situation opened the way to a large field of investigations, which led to the construction in the 1960's of the moduli spaces of *semi-stable* vector bundles of given rank and degree on $X$, achieved mainly by Mumford, Narasimhan and Seshadri. Ramanathan then extended this construction to prove the existence of moduli spaces for semi-stable principal $G$-bundles on $X$ for any connected reductive group $G$.

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These varieties, which will be denoted by $\mathcal{M}_G$ in this paper, have been intensively investigated since their construction, especially for $G = \mathbf{GL}_r$. They have more recently drawn new attention for the fundamental role they appeared to play in various subjects, like Conformal Field Theory or Langland's geometric correspondence.

In these notes we consider the following question:

If $H \longrightarrow G$ is a morphism between two reductive groups, what can we say about the induced morphism $\mathcal{M}_H \longrightarrow \mathcal{M}_G$ between moduli spaces?

This is a frequently encountered situation. For example, choosing for $H$ a maximal torus $T \simeq (\mathbf{G}_m)^l$ contained in $G$ gives a morphism from the moduli space $\mathcal{M}_T^0$ of topologically trivial $T$-bundles (which is isomorphic to $(J_X)^l$) to the variety $\mathcal{M}_G$. When $X$ is the projective line $\mathbf{P}^1$, we know from [13] that any principal $G$-bundle on $\mathbf{P}^1$ comes from a principal $T$-bundle. If $X$ is an elliptic curve, [17] shows that the morphism $\mathcal{M}_T^0 \longrightarrow \mathcal{M}_G$ is a finite morphism from $\mathcal{M}_T^0 \simeq X^l$ onto the connected component of $\mathcal{M}_G$ consisting of topologically trivial semi-stable $G$-bundles. For higher genus curves, let us just say that the morphism $\mathcal{M}^0_{\mathbf{G}_m} = J_X \longrightarrow \mathcal{M}_{\mathbf{SL}_2}$, which sends a line bundle $L$ to the vector bundle $L \oplus L^{-1}$, gives a beautiful way to investigate the geometry of the moduli spaces of semi-stable rank 2 vector bundles on $X$ (see [4]).

We study here the case of the classical groups $H = \mathbf{O}_r$ and $\mathbf{Sp}_{2r}$, naturally embedded in the general linear group. The moduli variety $\mathcal{M}_{\mathbf{O}_r}$ then parametrizes semi-stable orthogonal bundles $(E, q)$ of rank $r$ on $X$, and the morphism $\mathcal{M}_{\mathbf{O}_r} \longrightarrow \mathcal{M}_{\mathbf{GL}_r}$ just forgets the quadratic form $q$. In the same way, $\mathcal{M}_{\mathbf{Sp}_{2r}}$ parametrizes semi-stable symplectic bundles, and $\mathcal{M}_{\mathbf{Sp}_{2r}} \longrightarrow \mathcal{M}_{\mathbf{SL}_{2r}}$ forgets the symplectic form. We will also consider $\mathbf{SO}_r$-bundles, which are oriented orthogonal bundles $(E, q, \omega)$, that is orthogonal bundles $(E, q)$ together with an orientation, which is defined as a section $\omega$ in $H^0(X, \mathcal{O}_X)$ satisfying $\tilde{q}(\omega) = 1$ (where $\tilde{q}$ is the quadratic form on $\det E \simeq \mathcal{O}_X$ induced by $q$).

We have shown in [31] that the forgetful morphisms

$$\mathcal{M}_{\mathbf{O}_r} \longrightarrow \mathcal{M}_{\mathbf{GL}_r} \quad \text{and} \quad \mathcal{M}_{\mathbf{Sp}_{2r}} \longrightarrow \mathcal{M}_{\mathbf{SL}_{2r}}$$

are both closed immersions. In other words, these morphisms identify the varieties of semi-stable orthogonal and symplectic bundles with closed subschemes of the variety of all vector bundles. Note that this means that the images in $\mathcal{M}_{\mathbf{GL}_r}$ of these two forgetful morphisms are normal subschemes. The proof involves an infinitesimal study of these varieties, which naturally leads to some considerations coming from representation theory of quivers (for example, we use the fact that $\mathcal{M}_{\mathbf{GL}_r}$ is locally isomorphic to the variety parametrizing semi-simple representations of a given quiver). We present in Section 3 a proof of this result which simplifies a little the one given in [31].

The moduli spaces $\mathcal{M}_G$ are in general not regular (nor even locally factorial), and a basic question is to describe their singular locus and the nature of the singularities. If $X$ has genus $g \geqslant 2$, the singular locus of $\mathcal{M}_{\mathbf{SL}_r}$ has a nice description, which has been known for long (see [21]): a semi-stable vector bundle defines a smooth point in $\mathcal{M}_{\mathbf{SL}_r}$ if and only if it is a *stable* vector bundle, except when $r = 2$ and $g = 2$ (in this very particular case, $\mathcal{M}_{\mathbf{SL}_2}$ is isomorphic to $\mathbf{P}^3$). For $G$-bundles one has to consider *regularly stable* bundles, which are stable $G$-bundle $P$ whose automorphism group $\mathrm{Aut}_G(P)$ is equal to the center $Z(G)$ of $G$. Such a bundle defines a smooth point in $\mathcal{M}_G$, and one can expect the converse to hold, barring some particular cases.

We solve this question for classical groups. Using Schwarz's classification [30] of coregular representations, we prove in Section 4 that the smooth locus of $\mathcal{M}_{\mathbf{SO}_r}$ is exactly the regularly stable locus, except when $X$ has genus 2 and $r = 3$ or 4. For symplectic bundles we prove that the smooth locus of $\mathcal{M}_{\mathbf{Sp}_{2r}}$ is exactly the set of regularly stable symplectic bundles (for $r \geqslant 2$). This proof, which requires a precise description of bundles associated to points of the moduli spaces, cannot be extended to another group $G$ without a good understanding of the nature of these bundles.

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## 1. The moduli spaces $\mathcal{M}_G$

**1.1.** — Let $X$ be a smooth projective irreducible curve of genus $g \geqslant 1$, defined over an algebraically closed field of characteristic 0.

We can associate to $X$ its *Jacobian variety* $J_X$, which parametrizes line bundles of degree 0 on the curve. It is a projective variety, whose closed points correspond bijectively to isomorphism classes of degree 0 line bundles on $X$. Moreover, $J_X$ has the following *moduli property*:

- if $\mathcal{L}$ is a family of degree 0 line bundles on $X$ parametrized by a scheme $T$, the classifying map $\varphi$ which maps a point $t \in T$ on the point in $J_X$ associated to the line bundle $\mathcal{L}_t$ defines a morphism $\varphi\colon T \longrightarrow J_X$,
- $J_X$ is "universal" for this property.

We should also mention here that $J_X$ comes with a (non-unique) *Poincaré bundle* $\mathcal{P}$ on the product $J_X \times X$. It is a line bundle on $J_X \times X$, whose restriction $\mathcal{P}_a$ to $\{a\} \times X$ is exactly the line bundle associated to the point $a \in J_X$.

The Jacobian variety inherits many geometric properties from its moduli interpretation: let us just note here that it is an abelian variety which naturally carries a principal polarization. This extra data allows to describe sections of line bundles on $J_X$ in terms of *theta functions*. This analytical interpretation of geometric objects defined on $J_X$ provides a powerful tool to investigate the beautiful relations between the curve and its Jacobian.

**1.2.** — It has thus been natural to look for some possible generalizations of this situation. To do this, we can remark that line bundles are exactly principal $\mathbf{G}_m$-bundles. Replacing the multiplicative group $\mathbf{G}_m$ by any reductive group $G$ leads to the consideration of *principal $G$-bundles* on $X$.

When $G$ is the linear group $\mathbf{GL}_r$, they are vector bundles on $X$. Topologically, vector bundles on the curve $X$ are classified by their rank $r$ and degree $d$, and the natural question is to find an algebraic variety whose points correspond to isomorphism classes of vector bundles on $X$ of fixed rank and degree. The idea that such varieties parametrizing vector bundles should exist and give the desired non-abelian generalization of the Jacobian variety goes back to Weil (see [34]). However, the situation cannot be as simple as it is for line bundles. Indeed, the collection $\mathcal{V}_{r,d}$ of all vector bundles of rank $r$ and degree $d$ on $X$ is not *bounded*: we cannot find any family of vector bundles parametrized by a scheme $T$ such that every vector bundle in $\mathcal{V}_{r,d}$ appears in this family. So we need to exclude some bundles in order to have a chance to get a variety enjoying a relevant moduli property.

As we have said in the introduction, the construction of these moduli spaces of vector bundles on $X$ has been carried out in the 1960's, mainly by Mumford and by Narasimhan and Seshadri. They happened to show that one has to restrict to *semi-stable* bundles to obtain a reasonable moduli variety. This notion was introduced first by Mumford in [20] in the light of Geometric Invariant Theory.

Let us define the *slope* of a vector bundle $E$ as the ratio $\mu(E) = \deg(E)/\operatorname{rk}(E)$.

***Definition 1.3***. — A vector bundle $E$ on $X$ is said to be *stable* (resp. *semi-stable*) if we have, for any proper subbundle $F \subset E$, the slope inequality

$$\mu(F) < \mu(E) \quad (\text{resp. } \mu(F) \leqslant \mu(E)).$$

We will mainly be concerned in the following with degree 0 vector bundles. In this case, saying that a bundle is stable just means that it does not contain any subbundle of degree $\geqslant 0$.

Mumford's GIT allowed him to provide the set of isomorphism classes of stable bundles of given rank and degree with the structure of a quasi-projective variety.

***Theorem 1.4* (Mumford)**. — *There exists a coarse moduli scheme $\mathcal{U}_X^{\mathrm{st}}(r,d)$ for stable vector bundles of rank $r$ and degree $d$ on $X$. Its points correspond bijectively to isomorphism classes of stable bundles of rank $r$ and degree $d$.*

This result precisely means that, if $F^{\mathrm{st}}_{X,r,d}$ denotes the moduli functor which associates to a scheme $T$ the set of isomorphism classes of families of stable vector bundles of rank $r$ and degree $d$ on $X$ parametrized by $T$,

(i) there is a natural transformation $\varphi : F^{\mathrm{st}}_{X,r,d} \longrightarrow \operatorname{Hom}(-,\mathcal{U}_X^{\mathrm{st}}(r,d))$ such that any natural transformation $F^{\mathrm{st}}_{X,r,d} \longrightarrow \operatorname{Hom}(-,N)$ factors through a unique morphism $\mathcal{U}_X^{\mathrm{st}}(r,d) \longrightarrow N$,

(ii) the set of closed points of $\mathcal{U}_X^{\mathrm{st}}(r,d)$ is identified (via $\varphi$) to the set $F^{\mathrm{st}}_{X,r,d}(\operatorname{Spec} k)$ of isomorphism classes of stable vector bundles of rank $r$ and degree $d$.