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SOME REMARKS ON NAKAJIMA'S QUIVER VARIETIES OF TYPE A

by

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Abstract. — We try to clarify the relations between quiver varieties of type A and Kraft-Procesi proof of normality of nilpotent conjugacy classes closures.

***Résumé* (Quelques remarques sur les variétés de carquois de Nakajima, de type A)**
Nous essayons de clarifier les relations entre les variétés de carquois de type A et la preuve par Kraft et Procesi de la normalité des adhérences des classes de conjugaison de matrices nilpotentes.

1. Introduction

Kraft and Procesi proved in [3] that for any nilpotent $n \times n$ matrix A over an algebraically closed field $\mathbf{k}$ of characteristic zero, the closure $\overline{C_A}$ of the conjugacy class C_A of A is normal, Cohen-Macaulay with rational singularities. The main idea of the proof of this wonderful theorem is as follows: $\overline{C_A}$ is proved to be isomorphic to the categorical quotient for an affine variety Z of representations of a quiver with relations: $\overline{C_A} \cong Z/\!\!/H$, where H is a reductive group. Moreover, this Z is proved to be a reduced irreducible normal complete intersection, and this implies all the claimed properties of $\overline{C_A}$ as being inherited by the categorical quotients over reductive groups in general.

Nakajima in [5] and [6] introduced a setup related to the term *quiver variety.* A very particular case of that setup, when the underlying quiver is of type A and the additional vector spaces are of special dimension vector leads to the above variety Z used by Kraft and Procesi. Nakajima employed this observation in [5] to illustrate quiver varieties, in particular, he proved a nice theorem ([5, Theorem 7.3]) relating the quiver variety in this case with the cotangent bundle over a flag variety. The proof is based on another result ([5, Theorem 7.2]) that he claimed to be proved in [3]. Actually, that result was proved in [3, Proposition 3.4] only for special dimension vectors,

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not in the generality needed for Theorem 7.3. Unfortunately, this confusion haven't been corrected so far and we want to fill this gap, and without any contradiction with the valuable *sense* of Nakajima's result.

First of all, both Theorems 7.2 and 7.3 are true and we give proofs for them. In addition, we show that Theorem 7.3 is closely related with a result on Δ-filtered modules of Auslander algebra from [1]. On the other hand, the main part of the results of [3] (because [3, Proposition 3.4] is only a small part of these) can not be generalized, in particular, the variety Z can be reducible (see Example 4.3).

Our study does not claim to be a new result. Quite the contrary, we are trying to present the known results in their uncompromising beauty.

2. Kraft-Procesi setup and Nakajima's Theorem 7.2.

We present the setup used in [3] keeping the local notation. Consider a sequence of t vector spaces and linear mappings between them:

$$U_1 \underset{B_1}{\overset{A_1}{\rightleftarrows}} U_2 \underset{B_2}{\overset{A_2}{\rightleftarrows}} U_3 \cdots U_{t-1} \underset{B_{t-1}}{\overset{A_{t-1}}{\rightleftarrows}} U_t \tag{1}$$

Consider moreover the equations as follows:

$$B_1A_1 = 0; B_2A_2 = A_1B_1; B_3A_3 = A_2B_2; \cdots; B_{t-1}A_{t-1} = A_{t-2}B_{t-2} \tag{2}$$

and denote by Z the closed subvariety defined by these equations. The equations can be thought of as "commutativity" conditions for every $i = 2, \dots, t-1$: two possible compositions of $U_{i-1} \rightleftarrows U_i \rightleftarrows U_{i+1}$ yield the same endomorphism of U_i. The extra condition $B_1A_1 = 0$ combined with that commutativity implies $(A_1B_1)^2 = A_1(B_1A_1)B_1 = 0$. Inductively, we have for $i = 2, \dots, t-1$:

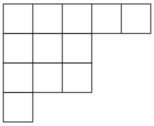
$$(B_iA_i)^i = (A_{i-1}B_{i-1})^i = A_{i-1}(B_{i-1}A_{i-1})^{i-1}B_{i-1} = 0 \Rightarrow (A_iB_i)^{i+1} = 0 \tag{3}$$

so all these endomorphisms are nilpotent. Denote $\dim U_i$ by n_i; so we have the dimension vector $(n_1, \dots, n_t)$. The variety Z is naturally acted upon by the group $G = GL_{n_1} \times \cdots \times GL_{n_t}$ and its normal subgroup $H = GL_{n_1} \times \cdots \times GL_{n_{t-1}}$. The above setup is interesting for any dimension vector but each of the texts [3] and [5, §7] considered those important for their purposes. Nakajima considered (in slightly different notation) *monotone* dimension vectors, that is, subject to the condition $n_1 < n_2 < \cdots < n_t$. One of the statements we feel necessary to clarify is the following (in our reformulation consitent with given notation):

***Theorem 2.1* (Theorem 7.2 from [5]).** — *Assume $(n_1, \dots, n_t)$ is monotone. Then the map $(A_1, B_1, \dots, A_{t-1}, B_{t-1}) \to A_{t-1}B_{t-1} : Z \to \mathrm{End}(U_t)$ is the categorical quotient with respect to H and the image is the conjugacy class closure for a nilpotent matrix.*

Instead of the proof for this Theorem it is stated in [5] that this result is proved in [3]. This is not true, because in [3] a smaller subset of dimensions was considered and the most part of the results concerns this subset, though the developed methods do allow to recover the proof of the above Theorem (see §4).

For the main goal of [3] it was sufficient to consider the dimensions as follows. Let $\eta = (p_1, p_2, \dots, p_k)$ be a partition with $p_1 \geq p_2 \geq \dots \geq p_k$. By $\hat{\eta} = (\hat{p}_1, \dots, \hat{p}_m)$ denote the *dual partition* such that $\hat{p}_i \doteq \#\{j | p_j \geq i\}$. In the Young diagram language, the diagram with *rows* consisting of $p_1, p_2, \dots, p_k$ boxes, respectively has *columns* consisting of $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_m$ boxes, respectively. For example, the dual partition to $\eta = (5, 3, 3, 1)$ is $\hat{\eta} = (4, 3, 3, 1, 1)$ as shows the Young diagram of η



Now, if $\eta = (p_1, p_2, \dots, p_k)$ is a partition such that $p_1 = t$ set

$$n_1 = \hat{p}_t; n_2 = \hat{p}_{t-1} + \hat{p}_t; \cdots; n_t = \hat{p}_1 + \hat{p}_2 + \cdots + \hat{p}_t. \tag{4}$$

So $n_1, \dots, n_t$ are the volumes of an increasing sequence of Young diagrams such that the previous diagram is the result of collapsing the first column of the next one. For example, the above partition yields the dimension vector $(1, 2, 5, 8, 12)$. This way we define a vector $n(\eta) = (n_1, \dots, n_t)$ and the set of all such vectors can be characterized by the inequalities as follows:

$$n_1 \leq n_2 - n_1 \leq n_3 - n_2 \leq \cdots \leq n_t - n_{t-1}. \tag{5}$$

In particular, this is a monotone sequence. Moreover, let C be the Cartan matrix of type A_{t-1} and set $v = (n_1, \dots, n_{t-1})$, $w = (0, \dots, 0, n_t)$. Then the formulae (5) are equivalent to

$$w - Cv \in \mathbf{Z}_+^{t-1} \tag{6}$$

Remark 2.1. — The condition (6) has a very important sense in Nakajima's theory. Namely, by [6, Proposition 10.5] it is equivalent to the set $\mathfrak{M}_0^{reg}(v, w)$ being nonempty, which means that the generic orbit in Z is closed with trivial stabilizer. The most interesting general Nakajima's results hold under this condition and in this particular case are just equivalent to what is proved in [3].

A partition $\eta = (p_1, \dots, p_k)$ of t yields a nilpotent conjugacy class C_η of matrices with Jordan blocks of size $p_1, \dots, p_k$, and moreover, a special matrix $A \in C_\eta$ such that basis vectors of $\mathbf{k}^t$ correspond to the boxes of Young diagram and A maps the boxes from the first column to 0 and each of the other boxes to its left neighbour.

We now state a result from [3], which is very close to Theorem 2.1. Actually our statement is more strong than in [3] but one can easily check that the original argument works for this statement without any change.

***Proposition 2.2* (Proposition 3.4 from [3]).** — **1.** *The map* $\Theta : Z \to \mathrm{End}(U_t)$, $\Theta(A_1, B_1, \ldots, A_{t-1}, B_{t-1}) = A_{t-1}B_{t-1}$ *is the categorical quotient with respect to* H *for arbitrary dimension vector* $(n_1, \ldots, n_t)$.

2. *If* $(n_1, \ldots, n_t) = n(\eta)$, *then the image of* Θ *is equal* $\overline{C_\eta}$.

3. Nakajima's Theorem 7.3

Before stating Nakajima's result we need some preliminary facts and notion. Let $(n_1, \ldots, n_t)$ be a monotone dimension vector. Denote by $\mathcal{F}$ the variety of partial flags $\{0\} = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{t-1} \subseteq E_t = \mathbf{k}^{n_t}$ with $\dim E_i = n_i$ for $i = 1, \ldots, t$. The variety $\mathcal{F}$ is projective and homogeneous with respect to the natural action of GL_{n_t}, $\mathcal{F} \cong GL_{n_t}/P$, where P is the stabilizer of a selected flag f_0, a parabolic subgroup in GL_{n_t}. Recall that the tangent space $T_{f_0} GL_{n_t}/P$ is isomorphic to $\mathfrak{p}_0^*$, where $\mathfrak{p}_0$ is the nilradical of the Lie algebra of P.

Consider a closed subset $X \subseteq \mathcal{F} \times \mathrm{End}(\mathbf{k}^{n_t})$ as follows:

$$X = \{(f, A) \in \mathcal{F} \times \mathrm{End}(\mathbf{k}^{n_t}) | AE_i \subseteq E_{i-1}, i = 1, \ldots, t\} \tag{7}$$

X is naturally isomorphic to the cotangent bundle $T^*\mathcal{F}$ because the fiber of the projection $p_1 : X \to \mathcal{F}$ over f_0 is $f_0 \times \mathfrak{p}_0$.

Let μ be the dual partition to the ordered sequence $(n_1, n_2 - n_1, \ldots, n_t - n_{t-1})$ (in particular, if $(n_1, \ldots, n_t) = n(\eta)$, then $\mu = \eta$). The following statement is well-known and can be found, e.g. in [2, Theorem 3.3]:

Proposition 3.1. — *We have* $p_2(X) = \overline{C_\mu}$.

Now we need to introduce shortly quiver varieties in this particular case. These are quotients by the action of a group, but two papers, [5] and [6] propose two different approachs to this notion, a Käler quotient and a quotient in the sense of Geometrical Invariant Theory, respectively. Though the results we discuss are in [5], we prefer the approach from [6].

Nakajima considered two quotiens of Z with respect to the action of H. The first, $\mathfrak{M}_0$ is just the categorical quotient, $\mathfrak{M}_0 = Z/\!/H$ so the geometrical points of $\mathfrak{M}_0$ are in 1-to-1 correspondance with the closed H-orbits in Z. On the other hand, one can consider the semi-stable locus $Z^{ss} \subseteq Z$ (actually with respect to a particular choice of a character of H but we consider just one as in [6]). It is proved in [6] in general case that Z^{ss} consists of stable points, that is, every H-orbit in Z^{ss} is closed in Z^{ss} and isomorphic to H. Hence, there is a *geometric quotient* $\mathfrak{M} = Z^{ss}/H$ (the construction of the quotient as an algebraic variety is usual for GIT, see [6, p.522]). In particular, the points of $\mathfrak{M}$ are in 1-to-1 correspondance with the H-orbits in Z^{ss}. Moreover, the categorical quotient $Z \to \mathfrak{M}_0$ gives rise to a natural map $\pi : \mathfrak{M} \to \mathfrak{M}_0$. Geometrically, π sends a stable orbit Hz to the unique closed (in Z) orbit in $\overline{Hz}$. Besides, the construction of $\mathfrak{M}$ implies that π is projective. Finally, Proposition 2.2 yields a convenient form of π as a map sending the stable orbit of $(A_1, B_1, \ldots, A_{t-1}, B_{t-1})$ to $A_{t-1}B_{t-1} \in \mathrm{End}(U_t)$.