Séminaires & Congrès

COLLECTION SMF

A SYMPLECTIC RESOLUTION FOR THE BINARY TETRAHEDRAL GROUP

Manfred Lehn & Christoph Sorger

GEOMETRIC METHODS IN REPRESENTATION THEORY, II

Numéro 25

Michel Brion, ed.

SOCIÉTE MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
25, 2010, p. 427–433

A SYMPLECTIC RESOLUTION FOR THE BINARY TETRAHEDRAL GROUP

by

Manfred Lehn & Christoph Sorger

Abstract. — We describe an explicit symplectic resolution for the quotient singularity arising from the four-dimensional symplectic representation of the binary tetrahedral group.

***Résumé* (Une résolution symplectique pour le groupe binaire tetrahedral).** — Nous décrivons une résolution symplectique explicite de la singularité quotient issue de la représentation symplectique de dimension quatre du groupe binaire tétraédral.

Let G be a finite group with a complex symplectic representation V. The symplectic form σ on V descends to a symplectic form $\bar{\sigma}$ on the open regular part of V/G. A proper morphism $f : Y \to V/G$ is a symplectic resolution if Y is smooth and if $f^*\bar{\sigma}$ extends to a symplectic form on Y. It turns out that symplectic resolutions of quotient singularities are a rare phenomenon. By a theorem of Verbitsky [14], a necessary condition for the existence of a symplectic resolution is that G be generated by symplectic reflections, i.e. by elements whose fix locus on V is a linear subspace of codimension 2. Given an arbitrary complex representation V_0 of a finite group G, we obtain a symplectic representation on $V_0 \oplus V_0^*$, where V_0^* denotes the contragradient representation of V_0. In this case, Verbitsky's theorem specialises to an earlier theorem of Kaledin [8]: For $V_0 \oplus V_0^*/G$ to admit a symplectic resolution, the action of G on V_0 should be generated by complex reflections, in other words, V_0/G should be smooth. The complex reflection groups have been classified by Shephard and Todd [13], the symplectic reflection groups by Cohen [3]. The list of Shephard and Todd contains as a sublist the finite Coxeter groups.

The question which of these groups $G \subset \mathrm{Sp}(V)$ admits a symplectic resolution for V/G has been solved for the Coxeter groups by Ginzburg and Kaledin [4] and for arbitrary complex reflection groups most recently by Bellamy [1]. His result is as follows:

2000 Mathematics Subject Classification. — 14B05; 14E15, 13P10.
Key words and phrases. — ???

© Séminaires et Congrès 25, SMF 2010

Version temporaire

Theorem 1. — (Bellamy) — *If $G \subset \mathrm{GL}(V_0)$ is a finite complex reflection group, then $V_0 \oplus V_0^*/G$ admits a symplectic resolution if and only if (G, V_0) belongs to the following cases:*

1. *$(S_n, \mathfrak{h})$, where the symmetric group S_n acts by permutations on the hyperplane $\mathfrak{h} = \{x \in \mathbb{C}^n \mid \sum_i x_i = 0\}$.*
2. *$((\mathbb{Z}/m)^n \rtimes S_n, \mathbb{C}^n)$, the action being given by multiplication with m-th roots of unity and permutations of the coordinates.*
3. *(T, S_1), where S_1 denotes a two-dimensional representation of the binary tetrahedral group T (see below).*

However, the technique of Ginzburg, Kaledin and Bellamy does not provide resolutions beyond the statement of existence. Case 1 corresponds to Coxeter groups of type A and Case 2 with $m = 2$ to Coxeter groups of type B. It is well-known that symplectic resolutions of $\mathfrak{h} \oplus \mathfrak{h}^*/S_n$ and $\mathbb{C}^n \oplus \mathbb{C}^n/(\mathbb{Z}/m)^n \rtimes S_n \cong \mathrm{Sym}^n(\mathbb{C}^2/(\mathbb{Z}/m))$ are given as follows:

For a smooth surface Y the Hilbert scheme $\mathrm{Hilb}^n(Y)$ of generalised n-tuples of points on Y provides a crepant resolution $\mathrm{Hilb}^n(Y) \to \mathrm{Sym}^n(Y)$. Applied to a minimal resolution of the A_{m-1}-singularity $\mathbb{C}^2/G$, $G \cong \mathbb{Z}/m$, this construction yields a small resolution $\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/G}) \to \mathrm{Sym}^n(\widetilde{\mathbb{C}^2/G}) \to \mathrm{Sym}^n(\mathbb{C}^2/G)$. Similarly, $(\mathfrak{h} \oplus \mathfrak{h}^*)/S_n$ is the fibre over the origin of the barycentric map $\mathrm{Sym}^n(\mathbb{C}^2) \to \mathbb{C}^2$. Thus $(\mathfrak{h} \oplus \mathfrak{h}^*)/S_n$ is resolved symplectically by the null-fibre of the morphism $\mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2) \to \mathbb{C}^2$.

It is the purpose of this note to describe an explicit symplectic resolution for the binary tetrahedral group.

1. The binary tetrahedral group

Let $T_0 \subset \mathrm{SO}(3)$ denote the symmetry group of a regular tetrahedron. The preimage of T_0 under the standard homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is the binary tetrahedral group T. As an abstract group, T is the semidirect product of the quaternion group $Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$ and the cyclic group $\mathbb{Z}/3$. As a subgroup of $\mathrm{SU}(2)$ it is generated by the elements

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \tau = -\frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$$

The binary tetrahedral group has 7 irreducible complex representations: A three-dimensional one arising from the quotient $T \to T_0 \subset \mathrm{SO}_3$, three one-dimensional representations $\mathbb{C}_j$ arising from the quotient $T \to \mathbb{Z}/3$ with τ acting by $e^{2\pi i j/3}$, and three two-dimensional representations S_0, S_1 and S_2. Here S_0 denotes the standard representation of T arising from the embedding $T \subset \mathrm{SU}_2$. This representation is symplectic, its quotient S_0/T being the well-known Klein-DuVal singularity of type E_6. The two other representations can be written as $S_j = S_0 \otimes \mathbb{C}_j$, $j = 1, 2$. They are dual to each other. It is as the subgroup of $\subset \mathrm{GL}(S_1)$ that T appears in the list

Version temporaire

of Shephard and Todd under the label "No. 4". The diagonal action of T on $S_1 \oplus S_2$ provides the embedding of T to Sp_4 that is of interest in our context.

Whereas the action of T on S_0 is symplectic, the action of T on S_1 and S_2 is generated by complex reflections of order 3. Overall, there are 8 elements of order 3 in T or rather 4 pairs of inverse elements, forming 2 conjugacy classes. To these correspond 4 lines in S_1 of points with nontrivial isotropy groups. Let $C_1 \subset S_1$ and $C_2 \subset S_2$ denote the union of these lines in each case. Then $C_1 \times S_2$ and $S_1 \times C_2$ are invariant divisors in $S_1 \oplus S_2$. However, the defining equations are invariant only up to a scalar. Consequently, their images W_1 and W_2 in the quotient $Z = S_1 \oplus S_2/T$ are Weil divisors but not Cartier. The reduced singular locus $\mathrm{sing}(Z)$ is irreducible and off the origin a transversal A_2 singularity. It forms one component of the intersection $W_1 \cap W_2$.

For $j = 1, 2$, let $\alpha_j : Z'_j \to Z$ denote the blow-up along W_j. Next, let W'_j be the reduced singular of locus Z'_j, and let $\beta_j : Z''_j \to Z'_j$ denote the blow-up along W'_j.

Theorem 2. — *The morphisms $\sigma_j = \alpha_j\beta_j : Z''_j \to Z$, $j = 1, 2$, are symplectic resolutions.*

Proof. — As all data are explicit, the assertion can be checked by brute calculation. To cope with the computational complexity we use the free computer algebra system SINGULAR [(1)] [5]. It suffices to treat one of the two cases of the theorem. We indicate the basic steps for $j = 2$. In order to improve the readability of the formulae we write $q = \sqrt{-3}$.

Let $\mathbb{C}[x_1, x_2, x_3, x_4]$ denote the ring of polynomial functions on $S_1 \oplus S_2$. The invariant subring $\mathbb{C}[x_1, x_2, x_3, x_4]^T$ is generated by eight elements, listed in table 1. The kernel I of the corresponding ring homomorphism

$$\mathbb{C}[z_1, \ldots, z_8] \to \mathbb{C}[x_1, x_2, x_3, x_4]^T$$

is generated by nine elements, listed in table 2. The curve C_2 is given by the semiinvariant $x_3^4 + 2qx_3^2x_4^2 + x_4^4$. In order to keep the calculation as simple as possible, the following observation is crucial: Modulo I, the Weil divisor W_2 can be described by 6 equations, listed in table 3. This leads to a comparatively 'small' embedding $Z'_2 \to \mathbb{P}^5_Z$ of Z-varieties. Off the origin, the effect of blowing-up of W_2 is easy to understand even without any calculation: the action of the quaternion normal subgroup $Q_8 \subset T$ on $S_1 \oplus S_2 \setminus \{0\}$ is free. The action of $\mathbb{Z}/3 = T/Q_8$ on $S_1 \oplus S_2/Q_8$ produces transversal A_2-singularities along a smooth two-dimensional subvariety. Blowing-up along W_1 or W_2 is a partial resolution: it introduces a $\mathbb{P}^1$ fibre over each singular point, and the total space contains a transversal A_1-singularity.

The homogeneous ideal $I'_2 \subset \mathbb{C}[z_1, \ldots, z_8, b_1, \ldots, b_6]$ that describes the subvariety $Z'_2 \subset \mathbb{P}^5_Z$ is generated by I and 39 additional polynomials. In order to understand the nature of the singularities of Z'_2 we consider the six affine charts $U_\ell = \{b_\ell = 1\}$. The

(1) A documented SINGULAR file containing all the calculations is available from the authors upon request.

Version température

Table 1: generators for the invariant subring $\mathbb{C}[x_1, x_2, x_3, x_4]^T$:

$$\begin{aligned}
z_1 &= x_1x_3 + x_2x_4, & z_4 &= x_2x_3^3 - qx_1x_3^2x_4 + qx_2x_3x_4^2 - x_1x_4^3,\\
z_2 &= x_3^4 - 2qx_3^2x_4^2 + x_4^4, & z_5 &= x_2^3x_3 - qx_1^2x_2x_3 + qx_1x_2^2x_4 - x_1^3x_4,\\
z_3 &= x_1^4 + 2qx_1^2x_2^2 + x_2^4, & z_6 &= x_1^5x_2 - x_1x_2^5,\\
z_7 &= x_3^5x_4 - x_3x_4^5 & z_8 &= x_1x_2^2x_3^3 - x_2^3x_3^2x_4 - x_1^3x_3x_4^2 + x_1^2x_2x_4^3.
\end{aligned}$$

Table 2: generators for $I = \ker(\mathbb{C}[z_1, \dots, z_8] \to \mathbb{C}[x_1, \dots, x_4]^T)$.

$$\begin{aligned}
&qz_1^3z_5 - z_1z_3z_4 - 2z_2z_6 - z_5z_8, && z_1z_5^2 + 2z_4z_6 + z_3z_8,\\
&qz_1^3z_4 + z_1z_2z_5 - 2z_3z_7 - z_4z_8, && z_1z_4^2 - 2z_5z_7 - z_2z_8,\\
&-z_1^4 + z_2z_3 - z_4z_5 - 3qz_1z_8, && qz_1^2z_3z_5 - 2z_1^3z_6 - z_3^2z_4 + z_5^3 - 6qz_6z_8,\\
&z_1^2z_4z_5 + qz_1^3z_8 + 4z_6z_7 - z_8^2, && qz_1^2z_2z_4 - 2z_1^3z_7 - z_4^3 + z_2^2z_5 - 6qz_7z_8,\\
&4z_1^2z_4z_5 + qz_3z_4^2 - qz_2z_5^2 + 4z_6z_7 + 8z_8^2
\end{aligned}$$

Table 3: generators for the ideal of the Weil divisor $W_2 \subset Z$.

$$\begin{aligned}
b_1 &= z_3z_7 + 2z_4z_8, & b_2 &= z_2z_4 + 2qz_1z_7, & b_3 &= z_2z_3 - 4qz_1z_8,\\
b_4 &= z_2^3 + 12qz_7^2, & b_5 &= z_1z_2^2 - 6z_4z_7, & b_6 &= z_1^2z_2 - qz_4^2.
\end{aligned}$$

Table 4: generators for the ideal sheaf J of $Z_2' \subset \mathbb{C}^7$ in the third chart:

$$\begin{aligned}
&4z_1b_1 + qz_3b_2 + z_5b_6, && z_1z_5 + z_3b_1 + qz_6b_6,\\
&z_1^2b_6 - z_3b_6^2 - 4qb_1^2 - 3z_5b_2, && z_1^2z_3 - z_3^2b_6 - qz_5^2 - 12z_6b_1,\\
&z_1^3 - z_1z_3b_6 + qz_5b_1 + 3qz_6b_2
\end{aligned}$$

result can be summarised like this: The singular locus of Z_2' is completely contained in $U_2 \cup U_3$, so only these charts are relevant for the discussion of the second blow-up. In fact, the corresponding affine coordinate rings have the following description:

$$R_2 = \mathbb{C}[z_1, b_3, b_4, b_5, b_6]/(b_5b_6 - 2qz_1)^2 + b_4(3qb_3 - b_6^3)$$

is a transversal A_1-singularity.

$$R_3 = \mathbb{C}[z_1, z_3, z_5, z_6, b_1, b_2, b_6]/J,$$

where J is generated by five elements, listed in table 4. Inspection of these generators shows that $\mathrm{Spec}(R_2)$ is isomorphic to the singularity $(\mathfrak{h}_3 \oplus \mathfrak{h}_3^*)/S_3$, the symplectic singularity of Coxeter type A_2 that appears as case 1 in Bellamy's theorem. It is well-known that blowing up the singular locus yields a small resolution. For arbitrary n, this is a theorem of Haiman [6, Prop. 2.6], in our case it is easier to do it directly. Thus blowing-up the reduced singular locus of Z_2' produces a smooth resolution $Z_2'' \to Z$.

It remains to check that the morphism $\alpha_2 : Z_2 \to Z$ is semi-small. For this it suffices to verify that the fibre $E = (\alpha_2^{-1}(0))_{\mathrm{red}}$ over the origin is two-dimensional and

Version temporaire