Séminaires & Congrès

COLLECTION SMF

MODULI OF LINEAR REPRESENTATIONS, SYMMETRIC PRODUCTS AND THE NON COMMUTATIVE HILBERT SCHEME

Francesco Vaccarino

GEOMETRIC METHODS IN REPRESENTATION THEORY, II

Numéro 25

Michel Brion, ed.

SOCIÉTE MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
25, 2010, p. 435–456

MODULI OF LINEAR REPRESENTATIONS, SYMMETRIC PRODUCTS AND THE NON COMMUTATIVE HILBERT SCHEME

by

Francesco Vaccarino

Abstract. — Let k be a commutative ring and let R be a commutative k-algebra. Let A be a R-algebra. We discuss the connections between the coarse moduli space of the n-dimensional representations of A, the non-commutative Hilbert scheme on A and the affine scheme which represents multiplicative homogeneous polynomial laws of degree n on A. We build a norm map which specializes to the Hilbert-Chow morphism on the geometric points when A is commutative and k is an algebraically closed field. This generalizes the construction done by Grothendieck, Deligne and others. When k is an infinite field and $A = k\{x_1, \ldots, x_m\}$ is the free k-associative algebra on m letters, we give a simple description of this norm map.

Résumé **(Modules de représentations linéaires, produits symétriques et le schéma non-commutatif de Hilbert)**

Soient k un anneau commutatif, R une k-algèbre commutative, et A une R-algèbre. Nous discutons les relations entre l'espace des modules grossier des représentations de dimension n de A, le schéma de Hilbert non commutatif de A, et le schéma affine qui représente les lois polynomiales multiplicatives homogènes de degré n sur A. Nous construisons une application norme, qui se spécialise en le morphisme de Hilbert-Chow sur les points géométriques lorsque A est commutative et k est un corps algébriquement clos. Ceci généralise une construction de Grothendieck, Deligne et autres. Lorsque k est un corps infini et $A = k\{x_1, \ldots, x_m\}$ est la k-algèbre associative libre sur m générateurs, nous donnons une description simple de l'application norme.

1. Introduction

Let k be a commutative ring and let R be a commutative k-algebra. Given a positive integer n and a R-algebra A one can consider three functors of points from the category $\mathcal{C}_R$ of commutative R-algebras to the small category of sets. All these functors are representable, namely

2000 Mathematics Subject Classification. — 14A15, 14C05, 16G99.

Key words and phrases. — Hilbert-Chow morphism, Hilbert schemes, linear representations, divided powers.

– $\mathcal{R}ep^n_A$ represents the functor induced by $B \to \hom_R(A, M_n(B))$, where $M_n(B)$ are the $n \times n$ matrices over B, for all $B \in \mathcal{C}_R$.
– the non-commutative Hilbert scheme Hilb^n_A represents the functor induced by

$$B \to \{ \text{ left ideals } I \subset A \otimes_k B \ : \ A \otimes_k B/I \text{ is a projective } R\text{-module of rank } n\}$$

for all $B \in \mathcal{C}_R$.
– Spec $\Gamma^n_R(A)^{ab}$ represents the functor induced by

$$B \to \{ \text{ multiplicative polynomial laws homogeneous of degree } n \text{ from } A \text{ to } B \ \}$$

for all $B \in \mathcal{C}_R$.

When A is commutative Hilb^A_n is the usual Hilbert scheme of n-points of $X = \operatorname{Spec} A$. A polynomial law is a kind of map generalizing polynomial mapping and coinciding with it over flat R-modules. The typical example of multiplicative polynomial law homogeneous of degree n is the determinant of $n \times n$ matrices. The R-algebra $\Gamma^n_R(A)^{ab}$ is the quotient of the R-algebra $\Gamma^n_R(A)$ of the divided powers of degree n on A by the ideal generated by commutators . When A is flat as R-module this coincides with the symmetric tensors of order n that is $\Gamma^n_R(A) \cong (A^{\otimes n})^{S_n}$, where S_n is the symmetric group. Therefore when A is commutative and flat we have $\operatorname{Spec} \Gamma^n_R(A)^{ab} \cong X^{(n)}$, the n-th symmetric product of $X = \operatorname{Spec} A$.

We discuss the connections between the coarse moduli space $\mathcal{R}ep^n_A//GL_n$ of the n-dimensional representations of A, the scheme Hilb^n_A and the affine scheme Spec $\Gamma^n_R(A)^{ab}$. We build a norm map from Hilb^n_A to $\Gamma^n_R(A)^{ab}$ which specializes to the Hilbert-Chow morphism on the geometric points when A is commutative and k is an algebraically closed field. This generalizes the construction done by Grothendieck, Deligne and others. When k is an infinite field and $A = k\{x_1, \ldots, x_m\}$ is the free k-associative algebra on m letters, we use the isomorphism Spec $\Gamma^n_k(A)^{ab} \cong \mathcal{R}ep^n_A//GL_n$ following from Theorem 1 in [26] to give a simple description of this norm map.

A field of application of our construction can be the extension to the positive characteristic case of some of the results due to L. Le Bruyn on noncommutative desingularization which can be found in [8]. Another intriguing possibility is to use our construction to extend the work done by C.H.Liu and S.T.Yau on D-Branes in [11] to the more recent non commutative case in [10].

This survey paper is based on [5, 25, 26, 27, 28].

Acknowledgements. — I would like to thank the organizers for their invitation. The author is supported by Progetto di Ricerca Nazionale COFIN 2007 "Teoria delle rappresentazioni: aspetti algebrici e geometrici".

Notations

Unless otherwise stated we adopt the following notations:
– k is a fixed commutative ground ring.
– R is a commutative k-algebra.

- B is a commutative R-algebra.
- A is an arbitrary R-algebra.
- $F = k\{x_1, \dots, x_m\}$ denotes the associative free k-algebra on m letters.
- $\mathcal{N}_-$, $\mathcal{C}_-$, Mod_- and Sets denote the categories of $-$algebras, commutative $-$algebras, $-$modules and sets, respectively.
- we write $\mathcal{A}(B, C)$ for the $\hom_{\mathcal{A}}(B, C)$ set in a category $\mathcal{A}$ with B, C objects in $\mathcal{A}$.

2. Moduli of representations

2.1. The universal representation. — We denote by $M_n(B)$ the full ring of $n \times n$ matrices over B. If $f : B \to C$ is a ring homomorphism we denote with $M_n(f) : M_n(B) \to M_n(C)$ the homomorphism induced on matrices.

Definition 2.1. — By an *n-dimensional representation of A over B* we mean a homomorphism of R-algebras $\rho : A \to M_n(B)$.

The assignment $B \to \mathcal{N}_R(A, M_n(B))$ defines a covariant functor $\mathcal{N}_R \to$ Sets. This functor is represented by a commutative R-algebra. We report here the proof of this fact to show how this algebra comes up using generic matrices. These objects will be also crucial in the construction of the norm map in Section 4.

Lemma 2.1. — [3, Lemma 1.2.] *For all $A \in \mathcal{N}_R$ there exist a commutative R-algebra $V_n(A)$ and a representation $\pi_A : A \to M_n(V_n(A))$ such that $\rho \mapsto M_n(\rho) \cdot \pi_A$ gives an isomorphism*

$$\mathcal{C}_R(V_n(A), B) \xrightarrow{\cong} \mathcal{N}_R(A, M_n(B)) \tag{1}$$

for all $B \in C_R$.

Proof. — Let A be an arbitrary R-algebra and suppose that there exist π_A and $V_n(A)$ as in the statement. Let $I \subset A$ be a bilateral ideal of A. Let J be the ideal generated by $\pi_A(I)$ in $M_n(V_n(A))$. There exists then an ideal $K \subset V_n(A)$ such that $J = M_n(K)$. It should be clear that $V_n(A/I) = V_n(A)/K$ and that $\pi_{A/I} : A/I \to M_n(V_n(A/I))$ is given by

$$\begin{array}{ccc} A & \xrightarrow{\pi_A} & M_n(V_n(A)) \\ \downarrow & & \downarrow \\ A/I & \xrightarrow[\pi_{A/I}]{} & M_n(V_n(A/I)) = M_n(V_n(A))/J \end{array}$$

It is enough now to prove the statement for $A = R\{x_1, \dots, x_m\} = R \otimes_k F$ the free associative R-algebra on m letters. Let $V_n(A) = R[\xi_{kij}]$ be the polynomial ring in variables $\{\xi_{kij} : i, j = 1, \dots, n,\ k = 1, \dots, m\}$ over the base ring R. To every n-dimensional representation of A over B it corresponds a unique m-tuple of $n \times n$ matrices, namely the images of $x_1, \dots, x_m$, hence a unique $\bar{\rho} \in \mathcal{C}_R(R[\xi_{kij}], B)$ such that $\bar{\rho}(\xi_{kij}) = (\rho(x_k))_{ij}$. Following C.Procesi [3, 16] we introduce the generic matrices. Let $\xi_k = (\xi_{kij})$ be the $n\times n$ matrix whose (i,j) entry is ξ_{kij} for $i,j = 1,\ldots,n$ and $k = 1,\ldots,m$. We call $\xi_1,\ldots,\xi_m$ the generic $n\times n$ matrices. Consider the map

$$\pi_A : A \to M_n(V_n(R)), \quad x_k \longmapsto \xi_k, \quad k = 1,\ldots,m\,.$$

It is then clear that the map $\mathcal{C}_R(V_n(A), B) \ni \sigma \mapsto M_n(\sigma)\cdot\pi_A \in \mathcal{N}_R(A, M_n(B))$ gives the isomorphism (1) in this case. □

Remark 2.1. — It should be clear that the number of generators m of A is immaterial and we can extend the above isomorphism to an arbitrary R-algebra.

Definition 2.2. — We write $\mathcal{R}ep_A^n$ to denote $\operatorname{Spec} V_n(A)$. It is considered as an R-scheme.
The map

$$\pi_A : A \to M_n(V_n(A)), \quad x_k \longmapsto \xi_k. \tag{2}$$

is called *the universal n-dimensional representation.*

Given a representation $\rho : A \to M_n(B)$ we denote by $\bar\rho$ its classifying map $\bar\rho : V_n(A) \to B$.

Example 2.1. — For the free algebra one has $\mathcal{R}ep_A^n \cong M_n^m$ the scheme whose B-points are the m-tuples of $n\times n$ matrices with entries in B.

Example 2.2. — Note that $\mathcal{R}ep_A^n$ could be quite complicated, as an example, when $A = \mathbb{C}[x,y]$ we obtain that $\mathcal{R}ep_A^n$ is the *commuting scheme* i.e. the couples of commuting matrices and it is not even known (but conjecturally true) if it is reduced or not, see [25].

2.2. GL_n-action

Definition 2.3. — We denote by GL_n the affine group scheme whose group of B-points form the group $\mathrm{GL}_n(B)$ of $n\times n$ invertible matrices with entries in B, for all $B\in\mathcal{C}_R$.

Define a GL_n-action on $\mathcal{R}ep_A^n$ as follows. For any $\varphi\in\mathcal{R}ep_A^n(B)$, $g\in\mathrm{GL}_n(B)$, let $\varphi^g : V_n(A)\to B$ be the R-algebra homomorphism corresponding to the representation given by

$$\begin{array}{ccc} A & \to & M_n(B) \\ a & \longmapsto & g(M_n(\varphi)\cdot\pi_A(a))g^{-1}. \end{array} \tag{3}$$

Note that if φ,φ' are B-points of $\mathcal{R}ep_A^n$, then the A-module structures induced on B^n by φ and φ' are isomorphic if and only if there exists $g\in\mathrm{GL}_n(B)$ such that $\varphi' = \varphi^g$.

Definition 2.4. — We denote by $\mathcal{R}ep_A^n//\mathrm{GL}_n = \operatorname{Spec} V_n(A)^{\mathrm{GL}_n(R)}$ the categorical quotient (in the category of R-schemes) of $\mathcal{R}ep_A^n$ by GL_n. It is the *(coarse) moduli space of n-dimensional linear representations* of A.