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THE PUNCTUAL HILBERT SCHEME: AN INTRODUCTION

by

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Abstract. — The punctual Hilbert scheme has been known since the early days of algebraic geometry in EGA style. Indeed it is a very particular case of the Grothendieck's Hilbert scheme which classifies the subschemes of projective space. The general Hilbert scheme is a key object in many geometric constructions, especially in moduli problems. The punctual Hilbert scheme which classifies the 0-dimensional subschemes of fixed degree, roughly finite sets of fat points, while being pathological in most settings, enjoys many interesting properties especially in dimensions at most three. Most interestingly it has been observed in this last decade that the punctual Hilbert scheme, or one of its relatives, the G-Hilbert scheme of Ito-Nakamura, is a convenient tool in many hot topics, as singularities of algebraic varieties, e.g McKay correspondence, enumerative geometry versus Gromov-Witten invariants, combinatorics and symmetric polynomials as in Haiman's work, geometric representation theory (the subject of this school) and many others topics.

The goal of these lectures is to give a self-contained and elementary study of the foundational aspects around the punctual Hilbert scheme, and then to focus on a selected choice of applications motivated by the subject of the summer school, the punctual Hilbert scheme of the affine plane, and an equivariant version of the punctual Hilbert scheme in connection with the A-D-E singularities. As a consequence of our choice some important aspects are not treated in these notes, mainly the cohomology theory, or Nakajima's theory. for which beautiful surveys are already available in the current litterature [20, 37, 42].

Papers with title *something an introduction* are often more difficult to read than *Lectures on something.* One can hope this paper is an exception. I would like to thank M. Brion for discussions and his generous help while preparing these notes.

Résumé. — ???

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1. Preliminary tools

The prototype of problems we are interested in is to describe in some sense the set of ideals of fixed codimension n in the polynomial ring in r variables $k[X_1, \dots, X_r]$ over a field k assumed algebraically closed to simplify.

In the one variable case, $k[X]$ being a principal ideal domain, an ideal I with $\dim k[X]/I = n$ is of the form $I = (P(X))$ with P monic and $\deg P = n$. These ideals are then parameterized by n parameters, the coefficients of P. In this case the punctual n-Hilbert scheme is an affine space $\mathbb{A}_k^n$. In a different direction, basic linear algebra tells us there is a precise relationship between on one hand the structure of the algebra $A = k[X]/(P)$ and on the other hand properties of the linear map $F \mapsto XF$ from A to A, summarized as follows

$P(X)$	A
without multiple factor	semi-simple
One root $\in k$ with multiplicity > 1	local, nilpotent
non zero discriminant	separable

One of our main goals in these lectures is to extend such a relationship to more general algebras than polynomials in one variable. One of our main theorems, in the two variables case, states that the set of all ideals with codimension n has a natural structure of a smooth algebraic variety of dimension $2n$. So to describe an ideal of codimension n in the polynomial ring $k[X, Y]$, we need exactly $2n$ parameters. Moreover the subset of ideals I with $k[X, Y]/I$ semi-simple is open and dense. The situation dramatically changes if the number of indeterminates is 3 or more. In any case the punctual Hilbert scheme appears to be a very amazing object.

Likewise, if A is a k-algebra (commutative throughout these notes, not necessarily of finite dimension as k-vector space) we can ask about the structure of the set of ideals of A. We shall see in case the dimension of A is finite, that the set of ideals $I \subset A$ with $\dim A/I = n$ is a projective variety, but infortunately in general, a very complicated one.

Throughout this text we fix an arbitrary base field k, not necessarily algebraically closed. In some cases however it will be convenient to assume $k = \overline{k}$, and sometimes the assumption of characteristic zero will be necessary. So in a first lecture the reader may assume $k = \overline{k}$ is a field of characteristic zero.

In this set of lectures, a scheme, or variety, will be mostly a k-scheme, that is a finite type scheme over k. Let us denote $\mathbf{Sch}_k$ the category of k-schemes, and correspondingly $\mathbf{Aff}_k$ the subcategory of affine k-schemes. One knows that $\mathbf{Aff}_k$ is the category opposite to the category $\mathbf{Alg}_k$ of commutative k-algebras of finite type. More generally $\mathbf{Sch}$ (resp. $\mathbf{Aff}$) stands for the category of (locally) noetherian schemes (resp. the category of affine noetherian schemes). If X is a scheme, $\mathbf{Aff}_X$ denotes the category of schemes over X, i.e. of schemes together with a morphism to X. For any $R \in \mathbf{Aff}$, $\operatorname{Spec} R$ stands for the spectrum of A, viewed as usual as a scheme. When $R = k[X_1, \ldots, X_n]/(F_1, \ldots, F_m)$ and $k = \overline{k}$, then $\operatorname{Spec} R \in \mathbf{Aff}_k$ can be thought of as the set $\{x \in k^n, F_1(x) = \cdots = F_m(x) = 0\}$ equipped with the ring of functions R. If X is a scheme, $\mathcal{O}_X$ stands for the sheaf of regular functions on open subsets of X. The stalk of $\mathcal{O}_X$ at a point x will be denoted $\mathcal{O}_{X,x}$ or $\mathcal{O}_x$ if X is fixed. By a point we always mean a closed point.

By an $\mathcal{O}_X$-module (resp. coherent module) we shall mean a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_X$-modules [31]. Finally a vector bundle, is a coherent $\mathcal{O}_X$-module which is locally free of rank n, i.e. at all $x \in X$ the stalk is a free $\mathcal{O}_{X,x}$- module of rank n. If $X = \operatorname{Spec} A$ the category of $\mathcal{O}_X$-modules is equivalent to the category of A-modules. A locally free module of rank n is a projective module of constant rank n.

We want to point out that the concept of flatness is essential to handle correctly families of objects in algebra or algebraic geometry, for us families of 0-dimensional subschemes, or ideals. We refer to [14], or [40] for the first definitions, and basic results.

Punctual Hilbert schemes will be obtained by glueing together affine schemes. This explains why the first section starts with some comments about this glueing process. Another basic operation that will be used in the sequel is the quotient of a scheme by a finite group action. This operation will be studied in detail in section 1.4.

1.1. Schemes versus representable functors

1.1.1. *Glueing affine schemes.* — One lesson of algebraic geometry in EGA style is that it is often better to think of a scheme $X \in \mathbf{Sch}$ as a contravariant functor, the so-called *functor of points*

$$\underline{X} : \mathbf{Sch} \to \mathbf{Ens} \quad (or,\ \mathbf{Aff} \to \mathbf{Ens}) \tag{1.1}$$

where $\underline{X}(S) = \operatorname{Hom}_{\mathbf{Sch}}(S, X)$. Essentially all the information about the scheme X can be read off the functor of points. It doesn't matter to choose either $\mathbf{Sch}$ or $\mathbf{Aff}$, indeed to reconstruct X from its functor of points, it is sufficient to know $\underline{X}$ on the subcategory $\mathbf{Aff}$. In this functorial setting a morphism $f : Y \to X$ can be thought of as a section $f \in \underline{X}(Y)$ or using Yoneda's lemma as a functorial morphism $\underline{Y} \to \underline{X}$. In the sequel we shall use the same letter to denote a scheme and its associated functor.

The functorial view-point as advocated before suggests that to construct a scheme, one has to identify first its functor of points $\mathcal{X}$, and then try to show that this functor is indeed the functors of points of a scheme. This last part which amounts to check $\mathcal{X}$ is representable, is in general not totally obvious. We must list the conditions about the functor $\mathcal{X} = \underline{X}$ expressing that X is the glueing of affine pieces. The first condition comes from restricting $\mathcal{X}$ to the category Open_S of open sets $U \subset S \in \mathbf{Sch}$, the morphisms being the inclusions $U \subset V$. The local character of morphisms implies that $\mathcal{X} : \mathrm{Open}_S \to \mathbf{Ens}$ is not only a presheaf but a Zariski sheaf. We say "Zariski" to keep in mind that the topology used to define the sheaf property is the Zariski topology. In other words if $S = \cup_i U_i$ is an open cover of $S \in \mathbf{Aff}$, the following diagram with obvious arrows is exact

$$h_X(S) \longrightarrow \prod_i h_X(U_i) \rightrightarrows \prod_{i,j} h_X(U_i \cap U_j) \tag{1.2}$$

Let $\mathcal{X}$ be a Zariski sheaf on $\mathbf{Aff}$. We say that $\mathcal{X}$ is *representable* if for some scheme X we have an isomorphism $\xi : \underline{X} \xrightarrow{\sim} \mathcal{X}$. As said before the Yoneda lemma asserts that such a morphism is determined by the single object $\xi(1_X) \in \mathcal{X}(X)$. It is convenient to identify ξ with this object and write $\xi : X \to \mathcal{X}$. In the same way let $F : \mathcal{X} \to \mathcal{Y}$ be a morphism. One says that F is representable if for all $\xi : S \to \mathcal{Y}$ the fiber product $\mathcal{X} \times_{\mathcal{Y}} S$, which is a sheaf, is representable.

If this is the case, F is said to be an open immersion (resp. closed immersion, a surjection) if for all ξ as above the projection $\mathcal{X} \times_{\mathcal{Y}} S \to S$ is an open immersion (resp. closed immersion, surjection). The following is the most naïve way to try to represent a functor, but it is sufficient for what follows.

Proposition 1.1. — *A Zariski sheaf $\mathcal{X}$ is representable, i.e a scheme X, if and only if:*

there exist a family morphisms $u_i : U_i \to \mathcal{X}$ such that the following conditions are satisfied

i) *for any i, $u_i : U_i \to \mathcal{X}$ is an open immersion, in particular $\coprod_i U_i \to \mathcal{X}$ is representable*
ii) *$u : U := \coprod_i U_i \to \mathcal{X}$ is surjective*
iii) *Finally X is separated (so really a scheme), if and only if the graph of the equivalence relation $U \times_{\mathcal{X}} U \hookrightarrow U \times U$ is a closed immersion.*

Proof. — First perform the fiber product

$$\begin{tikzcd} U_i \arrow[r, "u_i"] & \mathcal{X} \\ U_i \times_{\mathcal{X}} U_j \arrow[u, "v_i"] \arrow[r, "v_j"] & U_j \arrow[u, "u_j"'] \end{tikzcd}$$

so that condition ii) says $U_i \times_{\mathcal{X}} U_j$ is a scheme. Furthermore the arrows v_i, v_j are both open immersions. Let us denote $U_{i,j} \subset U_i$ and $U_{j,i} \subset U_j$ the corresponding open sets. The isomorphism $U_i \times_{\mathcal{X}} U_j \xrightarrow{\sim} U_{i,j}$ together with the corresponding one with U_{ji}, yields an isomorphism $\theta_{j,i}$, viz.

$$\begin{tikzcd} & U_i \times_{\mathcal{X}} U_j \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ U_{i,j} \arrow[rr, dotted, "\theta_{j,i}", "\sim"'] & & U_{j,i} \end{tikzcd} \tag{1.3}$$

The associativity of the fiber product quickly yields the following cocycle condition

$$\theta_{k,j|U_{j,i} \cap U_{j,k}} \theta_{j,i|U_{i,j} \cap U_{i,k}} = \theta_{k,i|U_{k,j} \cap U_{k,i}}, \quad \theta_{i,j}\theta_{j,i} = 1_{U_{i,j}} \tag{1.4}$$