Séminaires & Congrès

COLLECTION SMF

REPRESENTATIONS OF QUIVERS

Michel Brion

GEOMETRIC METHODS IN REPRESENTATION THEORY, I

Numéro 24

Michel Brion, ed.

SOCIÉTE MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
24, 2010, p. 101–142

REPRESENTATIONS OF QUIVERS

by

Michel Brion

Abstract. — These notes give an introduction to the theory of representations of quivers, in its algebraic and geometric aspects. The main result is Gabriel's theorem that characterizes quivers having only finitely many isomorphism classes of representations in any prescribed dimension.

Résumé. — Ces notes donnent une introduction à la théorie des représentations des carquois, sous ses aspects algébrique et géométrique. Le résultat principal est le théorème de Gabriel, qui caractérise les carquois dont les représentations de dimension donnée (arbitraire) forment un nombre fini de classes d'isomorphisme.

Introduction

Quivers are very simple mathematical objects: finite directed graphs. A representation of a quiver assigns a vector space to each vertex, and a linear map to each arrow. Quiver representations were originally introduced to treat problems of linear algebra, for example, the classification of tuples of subspaces of a prescribed vector space. But it soon turned out that quivers and their representations play an important role in representation theory of finite-dimensional algebras; they also occur in less expected domains of mathematics including Kac-Moody Lie algebras, quantum groups, Coxeter groups, and geometric invariant theory.

These notes present some fundamental results and examples of quiver representations, in their algebraic and geometric aspects. Our main goal is to give an account of a theorem of Gabriel characterizing quivers of finite orbit type, that is, having only finitely many isomorphism classes of representations in any prescribed dimension: such quivers are exactly the disjoint unions of Dynkin diagrams of types A_n, D_n, E_6, E_7, E_8, equipped with arbitrary orientations. Moreover, the isomorphism

2000 Mathematics Subject Classification. — 16G20; 14L30, 16G60.
Key words and phrases. — ???

classes of indecomposable representations correspond bijectively to the positive roots of the associated root system.

This beautiful result has many applications to problems of linear algebra. For example, when applied to an appropriate quiver of type D_4, it yields a classification of triples of subspaces of a prescribed vector space, by finitely many combinatorial invariants. The corresponding classification for quadruples of subspaces involves one-parameter families (the so-called tame case); for r-tuples with $r \geq 5$, one obtains families depending on an arbitrary number of parameters (the wild case).

Gabriel's theorem holds over an arbitrary field; in these notes, we only consider algebraically closed fields, in order to keep the prerequisites at a minimum. Section 1 is devoted to the algebraic aspects of quiver representations; it requires very little background. The geometric aspects are considered in Section 2, where familiarity with some affine algebraic geometry is assumed. Section 3, on representations of finitely generated algebras, is a bit more advanced, as it uses (and illustrates) basic notions of affine schemes. The reader will find more detailed outlines, prerequisites, and suggestions for further reading, at the beginning of each section.

Many important developments of quiver representations fall beyond the limited scope of these notes; among them, we mention Kac's far-reaching generalization of Gabriel's theorem (exposed in [11]), and the construction and study of moduli spaces (surveyed in the notes of Ginzburg, see also [15]).

Conventions. — Throughout these notes, we consider vector spaces, linear maps, algebras, over a fixed field k, assumed to be algebraically closed. All algebras are assumed to be associative, with unit; modules are understood to be left modules, unless otherwise stated.

1. Quiver representations: the algebraic approach

In this section, we present fundamental notions and results on representations of quivers and of finite-dimensional algebras.

Basic definitions concerning quivers and their representations are formulated in Subsection 1.1, and illustrated on three classes of examples. In particular, we define quivers of finite orbit type, and state their characterization in terms of Dynkin diagrams (Gabriel's theorem).

In Subsection 1.2, we define the quiver algebra, and identify its representations with those of the quiver. We also briefly consider quivers with relations.

The classes of simple, indecomposable, and projective representations are discussed in Subsection 1.3, in the general setting of representations of algebras. We illustrate these notions with results and examples from quiver algebras.

Subsection 1.4 is devoted to the standard resolutions of quiver representations, with applications to extensions and to the Euler and Tits forms.

The prerequisites are quite modest: basic material on rings and modules in Subsections 1.1-1.3; some homological algebra (projective resolutions, Ext groups, extensions) in Subsection 1.4.

We generally provide complete proofs, with the exception of some classical results for which we refer to [3]. Thereby, we make only the first steps in the representation theory of quivers and finite-dimensional algebras. The reader will find more complete expositions in the books [1, 2, 3] and in the notes [5]; the article [6] gives a nice overview of the subject.

1.1. Basic definitions and examples

DEFINITION 1.1.1. — A *quiver* is a finite directed graph, possibly with multiple arrows and loops. More specifically, a quiver is a quadruple

$$Q = (Q_0, Q_1, s, t),$$

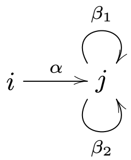
where Q_0, Q_1 are finite sets (the set of *vertices*, resp. *arrows*) and

$$s, t : Q_1 \longrightarrow Q_0$$

are maps assigning to each arrow its *source*, resp. *target*.

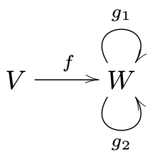
We shall denote the vertices by letters $i, j, \dots$ An arrow with source i and target j will be denoted by $\alpha : i \to j$, or by $i \xrightarrow{\alpha} j$ when depicting the quiver.

For example, the quiver with vertices i, j and arrows $\alpha : i \to j$ and $\beta_1, \beta_2 : j \to j$ is depicted as follows:



DEFINITION 1.1.2. — A *representation* M of a quiver Q consists of a family of vector spaces V_i indexed by the vertices $i \in Q_0$, together with a family of linear maps $f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ indexed by the arrows $\alpha \in Q_1$.

For example, a representation of the preceding quiver is just a diagram



where V, W are vector spaces, and f, g_1, g_2 are linear maps.

Definition 1.1.3. — Given two representations $M = \big((V_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1}\big)$, $N = (W_i, g_\alpha)$ of a quiver Q, a *morphism* $u : M \to N$ is a family of linear maps $(u_i : V_i \to W_i)_{i\in Q_0}$ such that the diagram

$$\begin{array}{ccc} V_{s(\alpha)} & \xrightarrow{f_\alpha} & V_{t(\alpha)} \\ {\scriptstyle u_{s(\alpha)}}\downarrow & & \downarrow{\scriptstyle u_{t(\alpha)}} \\ W_{s(\alpha)} & \xrightarrow{g_\alpha} & W_{t(\alpha)} \end{array}$$

commutes for any $\alpha \in Q_1$.

For any two morphisms $u : M \to N$ and $v : N \to P$, the family of compositions $(v_i u_i)_{i\in Q_0}$ is a morphism $vu : M \to P$. This defines the composition of morphisms, which is clearly associative and has identity elements $\mathrm{id}_M := (\mathrm{id}_{V_i})_{i\in Q_0}$. So we may consider the *category of representations of* Q, that we denote by $\mathrm{Rep}(Q)$.

Given two representations M, N as above, the set of all morphisms (of representations) from M to N is a subspace of $\prod_{i\in Q_0} \mathrm{Hom}(V_i, W_i)$; we denote that subspace by $\mathrm{Hom}_Q(M, N)$. If $M = N$, then

$$\mathrm{End}_Q(M) := \mathrm{Hom}_Q(M, M)$$

is a subalgebra of the product algebra $\prod_{i\in Q_0} \mathrm{End}(V_i)$.

Clearly, the composition of morphisms is bilinear; also, we may define direct sums and exact sequences of representations in an obvious way. In fact, one may check that $\mathrm{Rep}(Q)$ *is a k-linear abelian category*; this will also follow from the equivalence of $\mathrm{Rep}(Q)$ with the category of modules over the quiver algebra kQ, see Proposition 1.2.2 below.

Definition 1.1.4. — A representation $M = (V_i, f_\alpha)$ of Q is *finite-dimensional* if so are all the vector spaces V_i. Under that assumption, the family

$$\underline{\dim}\, M := (\dim V_i)_{i\in Q_0}$$

is the *dimension vector* of M; it lies in the additive group $\mathbb{Z}^{Q_0}$ consisting of all tuples of integers $\underline{n} = (n_i)_{i\in Q_0}$.

We denote by $(\varepsilon_i)_{i\in Q_0}$ the canonical basis of $\mathbb{Z}^{Q_0}$, so that $\underline{n} = \sum_{i\in Q_0} n_i\, \varepsilon_i$.

Note that every exact sequence of finite-dimensional representations

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

satisfies

$$\underline{\dim}\, M = \underline{\dim}\, M' + \underline{\dim}\, M''.$$

Also, any two isomorphic finite-dimensional representations have the same dimension vector. A central problem of quiver theory is *to describe the isomorphism classes of finite-dimensional representations of a prescribed quiver, having a prescribed dimension vector.*