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# LECTURES ON NAKAJIMA'S QUIVER VARIETIES

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# LECTURES ON NAKAJIMA'S QUIVER VARIETIES

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***Abstract.*** — In these lectures we define Nakajima quiver varieties, discuss their basic properties, and explain applications to the representation theory of Kac-Moody Lie algebras. We follow the original texts by Nakajima, with minor improvements. We provide some background concerning the notion of stability, GIT, and algebraic symplectic geometry. We also briefly discuss a few closely related topics, eg, the McKay correspondence.

***Résumé* (Notes sur les variétés de carquois de Nakajima).** — Dans ces notes, nous définissons les variétés de carquois de Nakajima, nous discutons leurs propriétés fondamentales, et nous expliquons leurs applications à la théorie des représentations des algèbres de Lie de Kac-Moody. Nous suivons les textes originaux de Nakajima, avec de petites améliorations. Nous présentons des rappels sur la notion de stabilité, la théorie géométrique des invariants, et la géométrie algébrique symplectique. Nous abordons aussi quelques sujets voisins, comme la correspondance de McKay.

## 1. Outline

**1.1. Introduction. —** Nakajima's quiver varieties are certain smooth (not necessarily affine) complex algebraic varieties associated with quivers. These varieties have been used by Nakajima to give a geometric construction of universal enveloping algebras of Kac-Moody Lie algebras (as well as a construction of quantized enveloping algebras for *affine* Lie algebras) and of all irreducible integrable (e.g., finite dimensional) representations of those algebras.

A connection between quiver representations and Kac-Moody Lie algebras has been first discovered by C. Ringel around 1990. Ringel produced a construction of $U_q(\mathfrak{n})$, the *positive part* of the quantized enveloping algebra $U_q(\mathfrak{g})$ of a Kac-Moody Lie algebra $\mathfrak{g}$, in terms of a *Hall algebra* associated with an appropriate quiver, cf. [49] for an exposition. Shortly afterwards, G. Lusztig combined Ringel's ideas with the powerful

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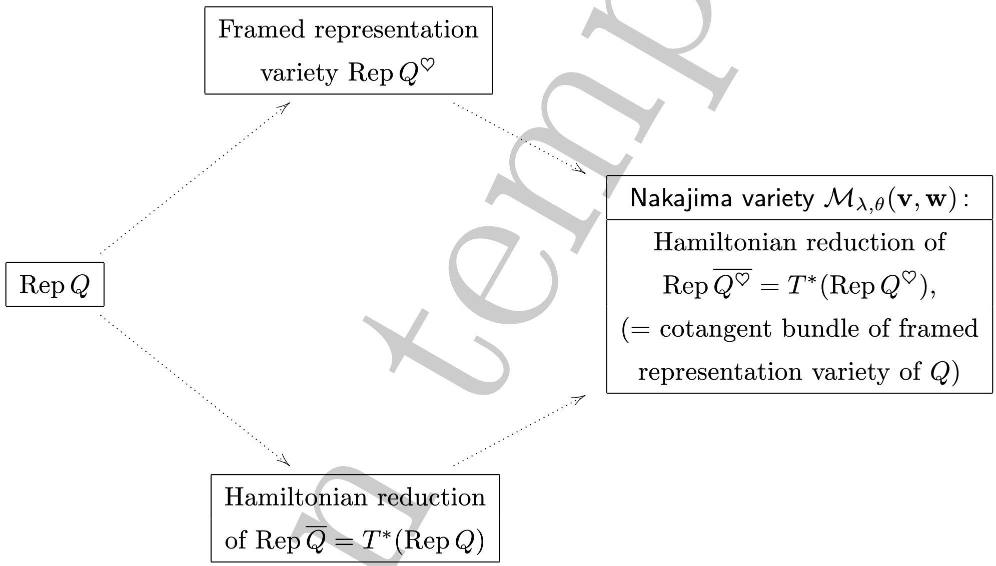
technique of perverse sheaves to construct a *canonical basis* of $U_q(\mathfrak{n})$, see [31, 32], and also [48].

The main advantage of Nakajima's approach (as opposed to the earlier one by Ringel and Lusztig) is that it yields a geometric construction of the whole algebra $U(\mathfrak{g})$ rather than its positive part. At the same time, it also provides a geometric construction of simple integrable $U(\mathfrak{g})$-modules. Nakajima's approach also yields a similar construction of the algebra $U_q(L\mathfrak{g})$ and its simple integrable representations, where $L\mathfrak{g}$ denotes the loop Lie algebra associated to $\mathfrak{g}$.[1]

There are several steps involved in the definition of Nakajima's quiver varieties. Given a quiver $Q$, one associates to it three other quivers, $Q^\heartsuit$, $\overline{Q}$, and $\overline{Q^\heartsuit}$, respectively. In terms of these quivers, various steps of the construction of Nakajima varieties may be illustrated schematically as follows




**1.2. Nakajima's varieties and symplectic algebraic geometry.** — Nakajima's varieties also provide an important large class of examples of algebraic symplectic manifolds with extremely nice properties and rich structure, interesting in their own right. To explain this, it is instructive to consider a more general setting as follows.

Let $X$ be a (possibly singular) affine variety equipped with an algebraic Poisson structure. In algebraic terms, this means that $\mathbb{C}[X]$, the coordinate ring of $X$, is

[1] Note however that, unlike the Ringel-Lusztig construction, the approach used by Nakajima does *not* provide a construction of the *quantized* enveloping algebra $U_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ itself. A similar situation holds in the case of Hecke algebras, where the *affine* Hecke algebra has a geometric interpretation in terms of equivariant $K$-theory, see [5, 25], while the Hecke algebra of a finite Weyl group does not seem to have such an interpretation.

equipped with a Poisson bracket $\{-,-\}$, that is, with a Lie bracket satisfying the Leibniz identity.

Recall that any smooth symplectic algebraic manifold carries a natural Poisson structure.

***Definition 1.2.1***. — *Let $X$ be an irreducible affine normal Poisson variety. A resolution of singularities $\pi : \widetilde{X} \to X$ is called a* symplectic resolution *of $X$ provided $\widetilde{X}$ is a smooth complex algebraic symplectic manifold (with algebraic symplectic 2-form) such that the pull-back morphism $\pi^* : \mathbb{C}[X] \to \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is a Poisson algebra morphism.*

Below, we will be interested in the case where the variety $X$ is equipped, in addition, with a $\mathbb{C}^\times$-action that rescales the Poisson bracket and contracts $X$ to a (unique) fixed point $o \in X$. Equivalently, this means that the coordinate ring of $X$ is equipped with a *nonnegative* grading $\mathbb{C}[X] = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}^k[X]$ such that $\mathbb{C}^k[X] = 0\,(\forall k < 0)$, and $\mathbb{C}^0[X] = \mathbb{C}$ and, in addition, there exists a (fixed) positive integer $m > 0$, such that one has

$$\{\mathbb{C}^i[X], \mathbb{C}^j[X]\} \subset \mathbb{C}^{i+j-m}[X], \quad \forall i, j \geq 0.$$

In this situation, given a symplectic resolution $\pi : \widetilde{X} \to X$, we call $\pi^{-1}(o)$, the fiber of $\pi$ over the $\mathbb{C}^\times$-fixed point $o \in X$, the *central fiber*.

Symplectic resolutions of a Poisson variety with a contracting $\mathbb{C}^\times$-action as above enjoy a number of very favorable properties:

1. The map $\pi : \widetilde{X} \to X$ is automatically *semismall* in the sense of Goresky-MacPherson, i.e. one has $\dim(\widetilde{X} \times_X \widetilde{X}) = \dim X$, cf. [22].
2. We have a Poisson algebra *isomorphism* $\pi^* : \mathbb{C}[X] \xrightarrow{\sim} \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$, moreover, $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ for all $i > 0$. The $\mathbb{C}^\times$-action on $X$ admits a canonical lift to an algebraic $\mathbb{C}^\times$-action on $\widetilde{X}$, see [22].
3. The Poisson variety $X$ is a union of *finitely many* symplectic leaves $X = \sqcup X_\alpha$, [23], and each symplectic leaf $X_\alpha$ is a locally closed smooth algebraic subvariety of $X$, [3].
4. For any $x \in X$, each rational homology group $H_\bullet(\pi^{-1}(x), \mathbb{Q})$ is generated by the fundamental classes of algebraic cycles, see [24].

   In particular, we have $H^{\mathrm{odd}}(\pi^{-1}(x), \mathbb{Q}) = 0$ and, for any $k \geq 0$, the cohomology group $H^{2k}(\pi^{-1}(x), \mathbb{C})$ has a pure Hodge structure of type $(k, k)$, cf. [10].
5. Each fiber of $\pi$, equipped with reduced scheme structure, is an isotropic subvariety of $\widetilde{X}$. The $\mathbb{C}^\times$-action provides a homotopy retraction of $\widetilde{X}$ to the central fiber $\pi^{-1}(o)$; in particular, we have $H^\bullet(\widetilde{X}, \mathbb{C}) \cong H^\bullet(\pi^{-1}(o), \mathbb{C})$.

The set $\widetilde{X} \times_X \widetilde{X}$ that appears in (i) may have several irreducible components and the semismallness property says that the dimension of any such component is $\leq \dim X$; in particular, the diagonal $X \subset \widetilde{X} \times_X \widetilde{X}$ is one such component of maximal dimension. The semismallness statement in (i) can be deduced from (v), using some elementary algebraic geometry, see [12] for a proof.

Let $K(Z)$ denote the Grothendieck group of the category of coherent sheaves on a scheme $Z$. We say that '*the scheme $Z$ has a decomposable diagonal in $K$-theory*' if there exist algebraic vector bundles $\mathcal{E}_i$, $\mathcal{F}_i$, $i = 1, \ldots, r$, on $Z$, such that, for the class $[\mathcal{O}_\Delta] \in K(Z \times Z)$ of the structure sheaf of the diagonal $Z \subset Z \times Z$, an equation

$$[\mathcal{O}_\Delta] = \sum_{i=1}^{n} (-1)^i \cdot [\mathcal{E}_i \boxtimes \mathcal{F}_i] \quad \text{holds in} \quad K(Z \times Z). \tag{1.2.2}$$

Essential parts of properties (ii) and (iv) above are based on the following theorem, to be proved in section 5.5 below.

***Theorem 1.2.3***. — *Let $\pi : \widetilde{X} \to X$ be a symplectic resolution with a contracting $\mathbb{C}^\times$-action, as above. Then, one has*

(1) $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ *for all* $i > 0$.

(2) *For any $x \in X$, the fiber $X_x$ is an isotropic subvariety of $\widetilde{X}$.*

(3) *The variety $\widetilde{X}$ has a decomposable diagonal in $K$-theory.*

Part (1) of the theorem is a special case of a well known result of Grauert-Riemenschnider [**20**]; the main idea of the proof of part (2) is due to Wiezerba [**53**] (extended and completed by Namikawa [**46**]); part (3) of the theorem was proved by Kaledin [**24**].

Applying the Chern character map $\mathbb{Q} \otimes_{\mathbb{Z}} K(\widetilde{X} \times \widetilde{X}) \to H^*(\widetilde{X} \times \widetilde{X}, \mathbb{Q})$ to equation (1.2.2), one can deduce that the groups $H_\bullet(\widetilde{X}, \mathbb{Q})$ are spanned by the Poincaré duals of the fundamental classes of algebraic cycles. This, combined with property (v) of symplectic resolutions, can be then used to prove property (iv) of symplectic resolutions in the special case of the central fiber.

**1.3.** — We discuss now several especially important examples of symplectic resolutions.

***Example 1.3.1* (Slodowy slices)**. — Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\langle e, h, f \rangle \subset \mathfrak{g}$ an $\mathfrak{sl}_2$-triple for a nilpotent element $e \in \mathfrak{g}$. Write $\mathfrak{z}_f$ for the centralizer of $f$ in $\mathfrak{g}$, and $\mathcal{N}$ for the *nilcone*, the subvariety of all nilpotent elements of $\mathfrak{g}$. Slodowy has shown that the intersection $\mathcal{S}_e := \mathcal{N} \cap (e + \mathfrak{z}_f)$ is reduced, normal, and that there is a $\mathbb{C}^\times$-action on $\mathcal{S}_e$ that contracts $\mathcal{S}_e$ to $e$, cf. eg. [5], §3.7 for an exposition.

The variety $\mathcal{S}_e$ is called the *Slodowy slice* for $e$ (the variety $\mathcal{S}_e$ has been known already to Harish-Chandra; it was studied in detail and extensively used by P. Slodowy [**51**]).

Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of the Killing form, and view $\mathcal{S}_e$ as a subvariety in $\mathfrak{g}^*$. Then, the standard Kirillov-Kostant Poisson structure on $\mathfrak{g}^*$ induces a Poisson structure on $\mathcal{S}_e$. The symplectic leaves in $\mathcal{S}_e$ are obtained by intersecting $e + \mathfrak{z}_f$ with the various nilpotent conjugacy classes in $\mathfrak{g}$.

Let $\mathcal{B}$ denote the flag variety for $\mathfrak{g}$, that is, the variety of all Borel subalgebras in $\mathfrak{g}$, and let $T^*\mathcal{B}$ be the cotangent bundle on $\mathcal{B}$. There is a standard resolution of