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HAIMAN'S WORK ON THE $n!$ THEOREM, AND BEYOND

by

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Abstract. — The $n!$ theorem asserts the existence of an exotic S_n-equivariant vector bundle on the Hilbert scheme of n-points on the plane. Striking consequences of this include the positivity of Macdonald polynomials of type A, a generalised McKay correspondence for nth symmetric product of the plane, and a description of the ring of diagonal coinvariants of S_n. We explain the origin of this theorem, outline its proof by Haiman and its consequences, and then survey some related open problems and generalisations.

***Résumé* (Travaux de Haiman sur le théorème $n!$ et au-delà).** — Le théorème $n!$ établit l'existence d'un fibré vectoriel S_n-équivariant exotique sur le schéma de Hilbert de n points du plan. Quelques-unes des conséquences remarquables sont la positivité des polynômes de Macdonald de type A, une correspondance de McKay généralisée pour le $n$ième produit symétrique du plan, et une description de l'anneau des coinvariants diagonaux de S_n. Nous expliquons l'origine de ce théorème; décrivons la preuve de Haiman et ses conséquences, et enfin exposons quelques problèmes ouverts liés et généralisations.

Introduction

In the late 1980's Macdonald introduced some remarkable symmetric functions which now bear his name. They depend on two parameters, t and q, and under various specialisations recover well-known symmetric functions that we have grown to love, including Hall-Littlewood functions, Jack functions, monomial symmetric functions, Schur functions. Based on empirical evidence, Macdonald conjectured several fundamental and non-obvious properties, including that when expressed in the

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Schur basis, the transition functions for his symmetric functions actually belong to $\mathbb{N}[q^{\pm 1}, t^{\pm 1}]$. This is called the *Macdonald positivity conjecture*. Such a result has predecessors for some of the above symmetric functions in fewer parameters, and is of interest because it suggests something is being counted, and even being counted with respect to a bigrading (to account for the t and q).

It is now known what is being counted (or better to say, we know one thing that is being counted by the Macdonald functions): the Macdonald functions count some bigraded copies of the regular representation of the symmetric group. But where do such representations come from? The symmetric group S_n acts naturally on a set of commuting variables $x_1, \ldots, x_n$, but such an action will only produce a grading (and indeed had been used in the study of Hall-Littlewood functions). To get the bigrading Garsia and Haiman introduced a second set of variables $y_1, \ldots, y_n$ and then proceeded to seek candidates for associated spaces that might produce the regular representation. They found some very natural spaces that, in low degree, did exactly what was required; they conjectured that in general these would produce the required realisation of Macdonald polynomials. Since this conjecture predicted that a space of polynomials (in $2n$ variables) carried the regular representation of S_n, it was known as the $n!$ *conjecture*. This conjecture became rather famous: it was easy to state, and attractive since it generalised many celebrated results from symmetric function theory, representation theory and geometry. On the other hand, having two sets of variables seemed to make things much more difficult. However, what made the conjecture *really* interesting was that thanks to Haiman and Procesi, it introduced a new object to the field, namely $\mathrm{Hilb}^n \mathbb{C}^2$, and consequently many new structures.

After a long battle, Haiman succeeded to confirm the $n!$ conjecture. He showed that bigraded S_n-equivariant components of special fibres of an exotic bundle on $\mathrm{Hilb}^n \mathbb{C}^2$—called the Procesi bundle—are being counted by Macdonald's polynomials. His work is a mixture of combinatorics, representation theory, algebraic geometry and homological algebra. The conjecture has inspired and fed into many other recent developments in algebra, combinatorics and geometry. These include the discovery of symplectic reflection algebras by Etingof-Ginzburg, the homological symplectic McKay correspondence of Bezrukavnikov-Kaledin, new combinatorial statistics for partitions attached to Dyck paths introduced by Haglund, Haiman, Loehr, Warrington and others.

In these lectures we will outline the whole story, but at a rather general level. There are already several excellent expository articles written on this topic by Haiman and available on his homepage. They contain varying levels of detail, but serve as wonderful guides to his two main papers on these topics, [12] and [13].

The search for further understanding of the spaces described by Macdonald polynomials goes on; exciting progress is mentioned towards the end of these lectures.

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Lecture 1

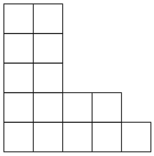
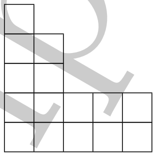
The best reference for much of the content in this lecture is *the* book [17].

1.1. Symmetric functions and the Frobenius map. — Recall that a sequence of integers $\boldsymbol{\lambda} = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0)$ is a partition of $|\lambda| = \sum \lambda_i$, written $\boldsymbol{\lambda} \vdash |\boldsymbol{\lambda}|$. We write $\ell(\boldsymbol{\lambda}) = r$ and set $n(\boldsymbol{\lambda}) = \sum_i (i-1)\lambda_i$. We let $\boldsymbol{\lambda}'$ denote the transpose of $\boldsymbol{\lambda}$. The dominance ordering on partitions is defined by

$$\boldsymbol{\lambda} \leq \boldsymbol{\mu} \text{ if and only if } \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \text{ for each } i > 0.$$

Note that $\boldsymbol{\lambda} \leq \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda}' \geq \boldsymbol{\mu}'$.

We identify a partition $\boldsymbol{\lambda}$ with its Young diagram $\boldsymbol{\lambda} = \{(p,q) \in \mathbb{N} \times \mathbb{N} : p < \lambda_{q+1}\}$. For example $\boldsymbol{\lambda} = (5,4,2,2,2)$ gives the following partitions of 15

$\boldsymbol{\lambda} \leftrightarrow$  $\quad \boldsymbol{\lambda}' \leftrightarrow$  .

Here $\boldsymbol{\lambda}' > \boldsymbol{\lambda}$.

Let Λ be the ring of symmetric functions, i.e.

$$\Lambda = \bigoplus_{k \geq 0} \varprojlim \mathbb{Q}[z_1, \ldots, z_n]_k^{S_n}.$$

These are functions of bounded degree, but in infinitely many variables $z = (z_1, z_2, \ldots)$. Later, we will extend scalars in Λ from $\mathbb{Q}$ to either $\mathbb{Q}(t)$ or $\mathbb{Q}(q,t)$. We will write Λ_t or $\Lambda_{q,t}$ respectively.

There are several natural bases for Λ, all indexed by partitions.

- *Power* $p_{\boldsymbol{\lambda}} = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_r}$ where $p_t = \sum_{i \geq 1} z_i^t$.
- *Monomial* $m_{\boldsymbol{\lambda}} = \sum_{\alpha \text{ permutation of } \boldsymbol{\lambda}} z^\alpha$ where if $\alpha = (\alpha_i)_{i \geq 1}$ then $z^\alpha = \prod_{i \geq 1} z_i^{\alpha_i}$.
- *Complete* $h_{\boldsymbol{\lambda}} = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_r}$ where $h_t = \sum_{|\boldsymbol{\mu}|=t} m_{\boldsymbol{\mu}}$.
- *Schur* $s_{\boldsymbol{\lambda}} = \det(z_i^{\lambda_j + n - j}) / \det(z_i^{n-j})$.

The last three bases are actually *integral*, meaning that they form bases for the ring of symmetric functions over $\mathbb{Z}$.

There is an inner product $\langle -, - \rangle$ on Λ, preserving degree. It is characterised by any of the following:

$$\begin{aligned} \langle s_{\boldsymbol{\lambda}}, s_{\boldsymbol{\mu}} \rangle &= \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}} \\ \langle p_{\boldsymbol{\lambda}}, p_{\boldsymbol{\mu}} \rangle &= \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}} z_{\boldsymbol{\lambda}} \\ \langle h_{\boldsymbol{\lambda}}, m_{\boldsymbol{\mu}} \rangle &= \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}. \end{aligned}$$

Here $z_{\boldsymbol{\lambda}} = \prod_{i \geq 1} i^{m_i} m_i!$ where $\boldsymbol{\lambda} = (1^{m_1}, 2^{m_2}, \ldots)$.

A good reason to care about symmetric functions is the following isometry of algebras, F, called the *Frobenius map*. To define it let $\operatorname{Rep}(S_n)$ denote the Grothendieck

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group of complex representations of S_n. Then $F : \bigoplus_{n\geq 0} \operatorname{Rep}(S_n) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \Lambda$ where $[A] \in \operatorname{Rep}(S_n)$ is sent to

$$F_A(z) := \frac{1}{n!} \sum_{w\in S_n} \chi^A(w) p_{\boldsymbol{\tau}(w)}(z)$$

where $\boldsymbol{\tau}(w)$ is the partition describing the cycle type of $w \in S_n$. On the left-hand-side we have an inner product given by the inner product on characters; multiplication is induced from

$$[A] \star [B] = [Ind_{S_n\times S_m}^{S_{n+m}}(A \boxtimes B)]$$

for $[A] \in \operatorname{Rep}(S_n), [B] \in \operatorname{Rep}(S_m)$. Here Ind denotes the induction functor, which for $H \leq G$ is defined on representations of H by $Ind_H^G(M) = \mathbb{C}G \otimes_{\mathbb{C}H} M$.

As an exercise you should show that $F_{\mathsf{triv}_n}(z) = h_n(z) = s_{(n)}(z)$ and that more generally $L_{\boldsymbol{\lambda}}$, the irreducible representation of S_n associated to $\boldsymbol{\lambda}$, is sent to $s_{\boldsymbol{\lambda}}(z)$, i.e. that $\chi^{\boldsymbol{\lambda}}(w) = \langle s_{\boldsymbol{\lambda}}, p_{\boldsymbol{\tau}(w)}\rangle$ (we write $\chi^{\boldsymbol{\lambda}}$ for $\chi^{L_{\boldsymbol{\lambda}}}$).

One consequence of this definition and these observations is that

$$F_{Ind_{S_{\boldsymbol{\mu}}}^{S_n}(\mathsf{triv}_{\boldsymbol{\mu}})}(z) = \prod_{i=1}^r F_{\mathsf{triv}_{\mu_i}}(z) = \prod_{i=1}^r h_{\mu_i} = h_{\boldsymbol{\mu}}$$

where $S_{\boldsymbol{\mu}} = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_r}$ is a Young subgroup of S_n. Thus

$$\langle s_{\boldsymbol{\lambda}}, h_{\boldsymbol{\mu}}\rangle = \langle \chi^{\boldsymbol{\lambda}}, Ind_{S_{\boldsymbol{\mu}}}^{S_n}(\mathsf{triv}_{\boldsymbol{\mu}})\rangle = K_{\boldsymbol{\lambda},\boldsymbol{\mu}},$$

a Kostka number (and in particular non-negative). So $h_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda}} K_{\boldsymbol{\lambda},\boldsymbol{\mu}} s_{\boldsymbol{\lambda}}$. Similarly, we see that $s_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu}} K_{\boldsymbol{\lambda},\boldsymbol{\mu}} m_{\boldsymbol{\mu}}$. Thus the Frobenius map gives positivity results (and interpretation) for transition matrices in symmetric function theory.

The Frobenius map generalises to a map on multi-graded S_n-representations. This means for instance that if $A = \bigoplus_{r,s\in\mathbb{Z}} A_{r,s}$ is a direct sum decomposition of S_n-representations labelled by pairs of integers, then we can set

$$F_A(z;q,t) = \sum_{r,s\in\mathbb{Z}} F_{A_{r,s}}(z) q^s t^r \in \Lambda_{q,t}.$$

So there is a relationship between bigraded representations of S_n and (q,t)-symmetric function theory.

1.2. Plethysm. — For any $A \in \Lambda_{q,t}$ we introduce the following $\mathbb{Q}(q,t)$-linear operation on $\Lambda_{q,t}$. For each $k > 0$ set $p_k[A] = A|_{q\mapsto q^k, t\mapsto t^k, z_i\mapsto z_i^k}$. Since the p_k freely generate $\Lambda_{q,t}$ as a $\mathbb{Q}(q,t)$-algebra this leads to an endomorphism $ev_A : \Lambda_{q,t} \longrightarrow \Lambda_{q,t}$ sending $p_{i_1}\cdots p_{i_t}$ to $p_{i_1}[A]\cdots p_{i_t}[A]$. This defines the *plethystic substitution* $f[A] = ev_A(f)$.

For instance, if we set $Z = z_1 + z_2 + \cdots = p_1 = h_1 = m_{(1)} = s_{(1)}$ then we see

$$p_k[Z] = p_k(z); \qquad p_k[-Z] = -p_k(z); \qquad \text{and so } p_k[Z(1-t)] = p_k(z)(1-t^k).$$

Similarly, using $Z/(1-t) = Z + tZ + t^2Z + \cdots$, we find

$$p_k[Z/(1-t)] = \sum_{i\geq 0} p_k(z)t^{ik} = \frac{1}{1-t^k} p_k(z).$$

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