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MOMENT GRAPHS AND REPRESENTATIONS

by

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Abstract. — Moment graphs and sheaves on moment graphs are basically combinatorial objects that have be used to describe equivariant intersectiion cohomology. In these lectures we are going to show that they can be used to provide a direct link from this cohomology to the representation theory of simple Lie algebras and of simple algebraic groups. The first section contains some background on equivariant cohomology.

***Résumé* (Graphes moment et représentations).** — Les graphes moment et les faisceaux sur ces graphes sont des objets de nature combinatoire, qui ont été utilisés pour déterminer la cohomologie d'intersection équivariante de certaines variétés. Dans ces notes, nous montrons comment ces objets permettent d'obtenir un lien direct entre cette cohomologie et la théorie des représentations des groupes et des algèbres de Lie simples. La première partie contient des résultats de base sur la cohomologie équivariante.

Introduction

In a 1979 paper Kazhdan and Lusztig introduced certain polynomials that nowadays are called Kazhdan-Lusztig polynomials. They conjectured that these polynomials determine the characters of infinite dimensional simple highest weight modules for complex semi-simple Lie algebras. Soon afterwards Lusztig made an analogous conjecture for the characters of irreducible representations of semi-simple algebraic groups in prime characteristics.

The characteristic 0 conjecture was proved within a few years. Concerning prime characteristics the best result known says that the conjecture holds in all characteristics p greater than an unknown bound depending on the type of the group.

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In both cases the proofs rely on the fact (proved by Kazhdan and Lusztig) that the Kazhdan-Lusztig polynomials describe the intersection cohomology of Schubert varieties. It was then quite complicated to link the representation theory to the intersection cohomology. In the characteristic 0 case this involved $\mathcal{D}$–modules and the Riemann-Hilbert correspondence. The proof of the weaker result in prime characteristics went via quantum groups and Kac-Moody Lie algebras.

In these notes I want to report on a more direct link between representations and cohomology. Most of this is due to Peter Fiebig. An essential tool is an alternative description of the intersection cohomology found by Tom Braden and Robert MacPherson. A crucial point is that on one hand one has to replace the usual intersection cohomology by *equivariant* intersection cohomology, while on the other hand one has to work with *deformations* of representations, i.e., with lifts of the modules to a suitable local ring that has our original ground field as its residue field.

Braden and MacPherson looked at varieties with an action of an (algebraic) torus; under certain assumptions (satisfied by Schubert varieties) they showed that the equivariant intersection cohomology is given by a combinatorially defined sheaf on a graph, the *moment graph* of the variety with the torus action.

Fiebig then constructed a functor from deformed representations to sheaves on a moment graph. This functor takes projective indecomposable modules to the sheaves defined by Braden and MacPherson. This is then the basis for a comparison between character formulae and intersection cohomology.

In Section 4 of these notes I describe Fiebig's construction in the characteristic 0 case. While Fiebig actually works with general (symmetrisable) Kac-Moody algebras, I have restricted myself here to the less complicated case of finite dimensional semisimple Lie algebras. The prime characteristic case is then discussed in Section 5, but with crucial proofs replaced by references to Fiebig's papers.

The two middle sections 2 and 3 discuss moment graphs and sheaves on them. I describe the Braden-MacPherson construction and follow Fiebig's approach to a localisation functor and its properties.

The first section looks at some cohomological background. A proof of the fact that the Braden-MacPherson sheaf describes the equivariant intersection cohomology was beyond the reach of these notes. Instead I go through the central definitions in equivariant cohomology and try to make it plausible that moment graphs have something to do with equivariant cohomology.

For advice on Section 1 I would like to thank Michel Brion and Jørgen Tornehave.

1. Cohomology

For general background in algebraic topology one may consult [14]. For more information on fibre bundles, see [17]. (I actually looked at the first edition published by McGraw-Hill.)

1.1. A simple calculation. — Consider the polynomial ring $S = k[x_1, x_2, x_3]$ in three indeterminates over a field k. Set $\alpha = x_1 - x_2$ and $\beta = x_2 - x_3$. Let us determine the following S–subalgebra of $S^3 = S \times S \times S$:

$$Z = \{\, (a, b, c) \in S^3 \mid a \equiv b \,(\mathrm{mod}\, S\alpha),\ b \equiv c \,(\mathrm{mod}\, S\beta),\ a \equiv c \,(\mathrm{mod}\, S(\alpha + \beta))\,\}. \quad (1)$$

We have clearly $(c, c, c) \in Z$ for all $c \in S$; it follows that $Z = S\,(1, 1, 1) \oplus Z'$ with

$$Z' = \{\, (a, b, 0) \in S^3 \mid a \equiv b \,(\mathrm{mod}\, S\alpha),\ b \in S\beta,\ a \in S(\alpha + \beta)\,\}.$$

Any triple $(b\,(\alpha + \beta), b\beta, 0)$ with $b \in S$ belongs to Z'. This yields $Z' = S\,(\alpha + \beta, \beta, 0) \oplus Z''$ where Z'' consists of all $(a, 0, 0)$ with $a \in S\,\alpha \cap S\,(\alpha + \beta)$. Since α and $\alpha + \beta$ are non-associated prime elements in the unique factorisation domain S, the last condition is equivalent to $a \in S\,\alpha\,(\alpha + \beta)$. So we get finally

$$Z = S\,(1, 1, 1) \oplus S\,(\alpha + \beta, \beta, 0) \oplus S\,(\alpha\,(\alpha + \beta), 0, 0). \quad (2)$$

So Z is a free S–module of rank 3.

Consider S as a graded ring with the usual grading doubled; so each x_i is homogeneous of degree 2. Then also S^3 and Z are naturally graded. Now (2) says that we have an isomorphism of graded S–modules

$$Z \simeq S \oplus S\langle 2\rangle \oplus S\langle 4\rangle \quad (3)$$

where quite generally $\langle n\rangle$ indicates a shift in the grading moving the homogeneous part of degree m into degree $n + m$.

The point about all this is that we have above calculated (in case $k = \mathbf{C}$) the equivariant cohomology $H^\bullet_T(\mathbf{P}^2(\mathbf{C}); \mathbf{C})$ where T is the algebraic torus $T = \mathbf{C}^\times \times \mathbf{C}^\times \times \mathbf{C}^\times$ acting on $\mathbf{P}^2(\mathbf{C})$ via $(t_1, t_2, t_3) \cdot [x : y : z] = [t_1 x : t_2 y : t_3 z]$ in homogeneous coordinates. Actually we have also calculated the ordinary cohomology $H^\bullet(\mathbf{P}^2(\mathbf{C}); \mathbf{C})$ that we get (in this case) as $Z/\mathfrak{m}Z$ where $\mathfrak{m}$ is the maximal ideal of S generated by the x_i, $1 \le i \le 3$. So we regain the well-known fact that $H^{2r}(\mathbf{P}^2(\mathbf{C}); \mathbf{C}) \simeq \mathbf{C}$ for $0 \le r \le 2$ while all remaining cohomology groups are 0.

1.2. Principal bundles. — Let G be a topological group. Recall that a *G–space* is a topological space X with a continuous action $G \times X \to X$ of G on X. If X is a G–space, then we denote by X/G the space of all orbits Gx with $x \in X$ endowed with the quotient topology: If $\pi \colon X \to X/G$ takes any $x \in X$ to its orbit Gx, then $U \subset X/G$ is open if and only if $\pi^{-1}(U)$ is open in X. It then follows that π is open since $\pi^{-1}(\pi(V)) = \bigcup_{g \in G} gV$ for any $V \subset X$.

A (numerable) *principal G–bundle* is a triple (E, p, B) where E is a G–space, B a topological space and $p \colon E \to B$ a continuous map such that there exists a numerable covering of B by open subsets U such that there exists a homeomorphism

$$\varphi_U \colon U \times G \to p^{-1}(U) \qquad \text{with } p \circ \varphi_U(u, g) = u \text{ and } \varphi_U(u, gh) = g\,\varphi_U(u, h) \quad (1)$$

for all $u \in U$ and $g, h \in G$. (The numerability condition is automatically satisfied if B is a paracompact Hausdorff space. We assume in the following all bundles to be numerable.)

Note that these conditions imply that the fibres of p are exactly the G–orbits on E, that each fibre $p^{-1}(b)$ with $b \in B$ is homeomorphic to G, and that G acts freely on E: If $g \in G$ and $x \in E$ with $g\,x = x$, then $g = 1$. It also follows that $Gx \mapsto p(x)$ is a homeomorphism from E/G onto B and that p is open.

For example the canonical map $p\colon \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{P}^n(\mathbf{C})$ is a principal bundle for the multiplicative group $\mathbf{C}^\times$. If we restrict p to the vectors of length 1, then we get a principal bundle $S^{2n+1} \to \mathbf{P}^n(\mathbf{C})$ for the group S^1 of complex numbers of length 1.

If G is a Lie group and H a closed Lie subgroup of G, then the canonical map $G \to G/H$ is a principal bundle for H acting on G by right multiplication. This is a fundamental result in Lie group theory.

If (E, p, B) is a principal bundle for a Lie group G and if H is a closed Lie subgroup of G, then $(E, \overline{p}, E/H)$ is a principal bundle for H where $\overline{p}\colon E \to E/H$ maps any $v \in E$ to its H-orbit Hv.

1.3. Universal principal bundles. — Let (E, p, B) be a principal bundle for a topological group G and let $f\colon B' \to B$ be a continuous map of topological spaces. Then one constructs an *induced* principal bundle $f^*(E, p, B) = (E', p', B')$: One takes E' as the fibre product

$$E' = B' \times_B E = \{\, (v, x) \in B' \times E \mid f(v) = p(x) \,\}$$

and one defines p' as the projection $p'(v, x) = v$. The action of G on E' is given by $g\,(v, x) = (v, gx)$; this makes sense as $p(gx) = p(x) = f(v)$. Consider an open subset U in B such that there exists a homeomorphism φ_U as in 1.2(1). Then $V := f^{-1}(U)$ is open in B', we have $(p')^{-1}(V) \subset V \times p^{-1}(U)$ and $\mathrm{id}_V \times \varphi_U$ induces a homeomorphism

$$\{\, (v, u, g) \in V \times U \times G \mid f(v) = u \,\} \longrightarrow (p')^{-1}(V),$$

hence using $(v, g) \mapsto (v, f(v), g)$ a homeomorphism $\psi_V : V \times G \to (p')^{-1}(V)$ satisfying $p' \circ \psi_V(v, g) = v$ and $g\,\psi_V(v, h) = \psi_V(v, gh)$ for all $v \in V$ and $g, h \in G$.

One can show: If $f_1\colon B' \to B$ and $f_2\colon B' \to B$ are homotopic continuous maps, then the induced principal bundles $f_1^*(E, p, B)$ and $f_2^*(E, p, B)$ are isomorphic over B'. Here two principal G–bundles (E_1, p_1, B) and (E_2, p_2, B) are called *isomorphic over* B if there exists a homeomorphism $\varphi\colon E_1 \to E_2$ with $p_2 \circ \varphi = p_1$ and $\varphi(g\,x) = g\,\varphi(x)$ for all $x \in E_1$.

A principal bundle (E_G, p_G, B_G) for a topological group G is called a *universal principal bundle* for G if for every principal G–bundle (E, p, B) there exists a continuous map $f\colon B \to B_G$ such that (E, p, B) is isomorphic to $f^*(E_G, p_G, B_G)$ over B and if f is uniquely determined up to homotopy by this property.

Milnor has given a general construction that associates to any topological group a universal principal bundle. A theorem of Dold (in *Ann. of Math.* **78** (1963), 223–255) says that a principal G–bundle (E, p, B) is universal if and only if E is contractible.

In case $G = S^1$ Milnor's construction leads to the following: Consider for any positive integer n the principal G–bundle $p_n\colon E_G^n = S^{2n+1} \to B_G^n = \mathbf{P}^n(\mathbf{C})$ as in 1.2. We have natural embeddings $E_G^n \to E_G^{n+1}$ and $B_G^n \to B_G^{n+1}$ induced by the embedding $\mathbf{C}^n \to \mathbf{C}^{n+1}$ mapping any $(x_1, x_2, \dots, x_n)$ to $(x_1, x_2, \dots, x_n, 0)$. These embeddings