Séminaires & Congrès

COLLECTION SMF

FOCK SPACE REPRESENTATIONS OF $U_q(\widehat{\mathfrak{sl}}_n)$

Bernard Leclerc

GEOMETRIC METHODS IN REPRESENTATION THEORY, I

Numéro 24

Michel Brion, ed.

SOCIÉTE MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
24, 2010, p. 319–361

FOCK SPACE REPRESENTATIONS OF $U_q(\widehat{\mathfrak{sl}}_n)$

by

Bernard Leclerc

Abstract. — We give an introduction to the Fock space representations of the affine Lie algebras $\widehat{\mathfrak{sl}}_n$ and their quantum analogues $U_q(\widehat{\mathfrak{sl}}_n)$. We explain the construction of their canonical bases, and the relationship with decomposition matrices of q-Schur algebras at an nth root of 1. In the last section we give a brief survey of some recent higher level analogues of these constructions.

Résumé. — Nous donnons une introduction aux représentations de Fock des algèbres de Lie affines $\widehat{\mathfrak{sl}}_n$ et de leurs analogues quantiques $U_q(\widehat{\mathfrak{sl}}_n)$. Nous expliquons la construction de leurs bases canoniques, et leur relation avec les matrices de décomposition des q-algèbres de Schur en une racine n-ième de l'unité. Dans la dernière partie nous donnons un bref compte-rendu de résultats analogues récents pour les niveaux supérieurs à 1.

1. Introduction

In the mathematical physics literature, the Fock space $\mathcal{F}$ is the carrier space of the natural irreducible representation of an infinite-dimensional Heisenberg Lie algebra $\mathfrak{H}$. Namely, $\mathcal{F}$ is the polynomial ring $\mathbb{C}[x_i \mid i \in \mathbb{N}^*]$, and $\mathfrak{H}$ is the Lie algebra generated by the derivations $\partial/\partial x_i$ and the operators of multiplication by x_i.

In the 70's it was realized that the Fock space could also give rise to interesting concrete realizations of highest weight representations of Kac-Moody affine Lie algebras $\widehat{\mathfrak{g}}$. Indeed $\widehat{\mathfrak{g}}$ has a natural Heisenberg subalgebra $\mathfrak{p}$ (the principal subalgebra) and the simplest highest weight $\widehat{\mathfrak{g}}$-module, called the basic representation of $\widehat{\mathfrak{g}}$, remains irreducible under restriction to $\mathfrak{p}$. Therefore, one can in principle extend the Fock space representation of $\mathfrak{p}$ to a Fock space representation $\mathcal{F}$ of $\widehat{\mathfrak{g}}$. This was first done for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ by Lepowsky and Wilson [37]. The Chevalley generators of $\widehat{\mathfrak{sl}}_2$ act on $\mathcal{F}$ via some interesting but complicated differential operators of infinite degree closely

2000 Mathematics Subject Classification. — 17B37, 17B67, 20C20.
Key words and phrases. — ???

related to the vertex operators invented by physicists in the theory of dual resonance models. Soon after, this construction was generalized to all affine Lie algebras $\widehat{\mathfrak{g}}$ of A, D, E type [23].

Independently and for different purposes (the theory of soliton equations) similar results were obtained by Date, Jimbo, Kashiwara and Miwa [4] for classical affine Lie algebras. Their approach is however different. They first endow $\mathcal{F}$ with an action of an infinite rank affine Lie algebra and then restrict it to various subalgebras $\widehat{\mathfrak{g}}$ to obtain their basic representations. In type A for example, they realize in $\mathcal{F}$ the basic representation of $\mathfrak{gl}_\infty$ (related to the KP-hierarchy of soliton equations) and restrict it to natural subalgebras isomorphic to $\widehat{\mathfrak{sl}}_n$ ($n \geqslant 2$) to get Fock space representations of these algebras (related to the KdV-hierarchy for $n = 2$). In this approach, the Fock space is rather the carrier space of the natural representation of an infinite-dimensional Clifford algebra, that is, an infinite dimensional analogue of an exterior algebra. The natural isomorphism between this "fermionic" construction and the previous "bosonic" construction is called the boson-fermion correspondence.

The basic representation of $\widehat{\mathfrak{g}}$ has level one. Higher level irreducible representations can also be constructed as subrepresentations of higher level Fock space representations of $\widehat{\mathfrak{g}}$ [11, 12].

After quantum enveloping algebras of Kac-Moody algebras were invented by Jimbo and Drinfeld, it became a natural question to construct the q-analogues of the above Fock space representations. The first results in this direction were obtained by Hayashi [16]. His construction was soon developed by Misra and Miwa [43], who showed that the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$ has a crystal basis (crystal bases had just been introduced by Kashiwara) and described it completely in terms of Young diagrams. This was the first example of a crystal basis of an infinite-dimensional representation. Another construction of the level one Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$ was given by Kashiwara, Miwa and Stern [29], in terms of semi-infinite q-wedges. This relied on the polynomial tensor representations of $U_q(\widehat{\mathfrak{sl}}_n)$ which give rise to the quantum affine analogue of the Schur-Weyl duality obtained by Ginzburg, Reshetikhin and Vasserot [13], and Chari and Pressley [3] independently.

In [32] and [35], some conjectures were formulated relating the decomposition matrices of type A Hecke algebras and q-Schur algebras at an nth root of unity on the one hand, and the global crystal basis of the Fock space representation $\mathcal{F}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ on the other hand. Note that [35] contains in particular the definition of the global basis of $\mathcal{F}$, which does not follow from the general theory of Kashiwara or Lusztig. The conjecture on Hecke algebras was proved by Ariki [1], and the conjecture on Schur algebras by Varagnolo and Vasserot [51].

Slightly after, Uglov gave a remarkable generalization of the results of [29], [35], and [51] to higher levels. Together with Takemura [48], he introduced a semi-infinite wedge realization of the level ℓ Fock space representations of $U_q(\widehat{\mathfrak{sl}}_n)$, and in [49, 50] he constructed their canonical bases and expressed their coefficients in terms of Kazhdan-Lusztig polynomials for the affine symmetric groups.

A full understanding of these coefficients as decomposition numbers is still missing. Recently, Yvonne [52] has formulated a precise conjecture stating that, under certain conditions on the components of the multi-charge of the Fock space, the coefficients of Uglov's bases should give the decomposition numbers of the cyclotomic q-Schur algebras of Dipper, James and Mathas [7]. Rouquier [45] has generalized this conjecture to all multi-charges. In his version the cyclotomic q-Schur algebras are replaced by some quasi-hereditary algebras arising from the category $\mathcal{O}$ of the rational Cherednik algebras attached to complex reflection groups of type $G(\ell, 1, m)$.

In these lectures we first present in Section 2 the Fock space representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$. We chose the most suitable construction for our purpose of q-deformation, namely, we realize $\mathcal{F}$ as a space of semi-infinite wedges (the fermionic picture). In Section 3 we explain the level one Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$ and construct its canonical bases. In Section 4 we explain the conjecture of [35] and its proof by Varagnolo and Vasserot. Finally, in Section 5 we indicate the main lines of Uglov's construction of higher level Fock space representations of $U_q(\widehat{\mathfrak{sl}}_n)$, and of their canonical bases, and we give a short review of Yvonne's work.

2. Fock space representations of $\widehat{\mathfrak{sl}}_n$

2.1. The Lie algebra $\widehat{\mathfrak{sl}}_n$ and its wedge space representations. — We fix an integer $n \geqslant 2$.

2.1.1. *The Lie algebra $\mathfrak{sl}_n$.* — The Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ of traceless $n \times n$ complex matrices has Chevalley generators

$$E_i = E_{i,i+1}, \quad F_i = E_{i+1,i}, \quad H_i = E_{ii} - E_{i+1,i+1}, \qquad (1 \leqslant i \leqslant n-1).$$

Its natural action on $V = \mathbb{C}^n = \oplus_{i=1}^n \mathbb{C} v_i$ is

$$E_i v_j = \delta_{j,i+1} v_i, \quad F_i v_j = \delta_{j,i} v_{i+1}, \quad H_i v_j = \delta_{j,i} v_i - \delta_{j,i+1} v_{i+1}, \qquad (1 \leqslant i \leqslant n-1).$$

We may picture the action of $\mathfrak{g}$ on V as follows

$$v_1 \xrightarrow{F_1} v_2 \xrightarrow{F_2} \cdots \xrightarrow{F_{n-1}} v_n$$

2.1.2. *The Lie algebra $L(\mathfrak{sl}_n)$.* — The loop space $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ is a Lie algebra under the Lie bracket

$$[a \otimes z^k, b \otimes z^l] = [a, b] \otimes z^{k+l}, \qquad (a, b \in \mathfrak{g}, \; k, l \in \mathbb{Z}).$$

The loop algebra $L(\mathfrak{g})$ naturally acts on $V(z) = V \otimes \mathbb{C}[z, z^{-1}]$ by

$$(a \otimes z^k) \cdot (v \otimes z^l) = av \otimes z^{k+l}, \qquad (a \in \mathfrak{g}, \; v \in V, \; k, l \in \mathbb{Z}).$$

2.1.3. *The Lie algebra* $\widehat{\mathfrak{sl}}_n$. — The affine Lie algebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_n$ is the central extension $L(\mathfrak{g}) \oplus \mathbb{C}c$ with Lie bracket

$$[a\otimes z^k+\lambda c,\ b\otimes z^l+\mu c] = [a,b]\otimes z^{k+l}+k\,\delta_{k,-l}\,\mathrm{tr}\,(ab)\,c, \qquad (a,b\in\mathfrak{g},\ \lambda,\mu\in\mathbb{C},\ k,l\in\mathbb{Z}).$$

This is a Kac-Moody algebra of type $A_{n-1}^{(1)}$ with Chevalley generators

$$e_i = E_i\otimes 1,\quad f_i = F_i\otimes 1,\quad h_i = H_i\otimes 1, \qquad (1\leqslant i\leqslant n-1),$$

$$e_0 = E_{n1}\otimes z,\quad f_0 = E_{1n}\otimes z^{-1},\quad h_0 = (E_{nn}-E_{11})\otimes 1 + c.$$

We denote by Λ_i $(i=0,1,\ldots,n-1)$ the fundamental weights of $\widehat{\mathfrak{g}}$. By definition, they satisfy

$$\Lambda_i(h_j) = \delta_{ij}, \qquad (0\leqslant i,j\leqslant n-1).$$

Let $V(\Lambda)$ be the irreducible $\widehat{\mathfrak{g}}$-module with highest weight Λ [22, §9.10]. If $\Lambda = \sum_i a_i\Lambda_i$ then the central element $c=\sum_i h_i$ acts as $\sum_i a_i\mathrm{Id}$ on $V(\Lambda)$, and we call $\ell = \sum_i a_i$ the level of $V(\Lambda)$. More generally, a representation V of $\widehat{\mathfrak{g}}$ is said to have *level* ℓ if c acts on V by multiplication by ℓ.

The loop representation $V(z)$ can also be regarded as a representation of $\widehat{\mathfrak{g}}$, in which c acts trivially. Define

$$u_{i-nk} = v_i\otimes z^k, \qquad (1\leqslant i\leqslant n,\ k\in\mathbb{Z}).$$

Then $(u_j \mid j\in\mathbb{Z})$ is a $\mathbb{C}$-basis of $V(z)$. We may picture the action of $\widehat{\mathfrak{g}}$ on $V(z)$ as follows

$$\cdots \xrightarrow{f_{n-2}} u_{-1} \xrightarrow{f_{n-1}} u_0 \xrightarrow{f_0} u_1 \xrightarrow{f_1} u_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} u_n \xrightarrow{f_0} u_{n+1} \xrightarrow{f_1} u_{n+2} \xrightarrow{f_2} \cdots$$

Note that this is not a highest weight representation.

2.1.4. *The tensor representations.* — For $r\in\mathbb{N}^*$, we consider the tensor space $V(z)^{\otimes r}$. The Lie algebra $\widehat{\mathfrak{g}}$ acts by derivations on the tensor algebra of $V(z)$. This induces an action on each tensor power $V(z)^{\otimes r}$, namely,

$$x(u_{i_1}\otimes\cdots\otimes u_{i_r}) = (xu_{i_1})\otimes\cdots\otimes u_{i_r}+\cdots+u_{i_1}\otimes\cdots\otimes(xu_{i_r}), \qquad (x\in\widehat{\mathfrak{g}},\ i_1,\cdots,i_r\in\mathbb{Z}).$$

Again c acts trivially on $V(z)^{\otimes r}$.

We have a vector space isomorphism $V^{\otimes r}\otimes\mathbb{C}[z_1^{\pm},\ldots,z_r^{\pm}] \xrightarrow{\sim} V(z)^{\otimes r}$ given by

$$(v_{i_1}\otimes\cdots\otimes v_{i_r})\otimes z_1^{j_1}\cdots z_r^{j_r} \mapsto (v_{i_1}\otimes z^{j_1})\otimes\cdots\otimes(v_{i_r}\otimes z^{j_r}), \qquad (1\leqslant i_1,\ldots,i_r\leqslant n,\ j_1,\ldots,j_r\in\mathbb{Z}).$$

2.1.5. *Action of the affine symmetric group.* — The symmetric group $\mathfrak{S}_r$ acts on $V^{\otimes r}\otimes\mathbb{C}[z_1^{\pm},\ldots,z_r^{\pm}]$ by

$$\sigma(v_{i_1}\otimes\cdots\otimes v_{i_r})\otimes z_1^{j_1}\cdots z_r^{j_r} = (v_{i_{\sigma^{-1}(1)}}\otimes\cdots\otimes v_{i_{\sigma^{-1}(r)}})\otimes z_1^{j_{\sigma^{-1}(1)}}\cdots z_r^{j_{\sigma^{-1}(r)}}, \qquad (\sigma\in\mathfrak{S}_r).$$

Moreover the abelian group $\mathbb{Z}^r$ acts on this space, namely $(k_1,\ldots,k_r)\in\mathbb{Z}^r$ acts by multiplication by $z_1^{k_1}\cdots z_r^{k_r}$. Hence we get an action on $V(z)^{\otimes r}$ of the affine symmetric group $\widehat{\mathfrak{S}}_r := \mathfrak{S}_r\ltimes\mathbb{Z}^r$. Clearly, this action commutes with the action of $\widehat{\mathfrak{g}}$.

It is convenient to describe this action in terms of the basis

$$(u_{\mathbf{i}} = u_{i_1}\otimes\ \cdots\otimes u_{i_r} \mid \mathbf{i} = (i_1,\ldots,i_r)\in\mathbb{Z}^r).$$