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SEMIGROUPS AND CONTROL THEORY

by

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Abstract. — The theory of strongly continuous semigroups, beyond the functional analysis, has found applications in many branches of mathematics such as differential equations, probability, geometry of Banach spaces and control theory. In this work we have limited ourselves to the study of the strictly necessary relationship between strongly continuous semigroups and the structure of the underlying Banach spaces for the study of the theory of control. Particularly important are the relationships that relate to the adjoint of a strongy continous semigroup that can be factorized through an Asplund space, in this case the adjoint semigroup is strongy continuous on $(0,\infty)$, a fact which is very important in control theory. For the case in which the semigroup in consideration is compact, the associated control system can never be exactly controllable in finite time.

***Résumé* (Semigroupes et théorie de contrôle).** — La théorie des semigroupes fortement continus, au-delà de lánalyse fonctionnelle, a trouvé des applications dans de nombreuses branches des mathématiques comme les équations différentielles, les probabilités, la géométrie des espaces de Banach et la théorie du contrôle.

Dans ce travail, nous nous sommes limités à l'étude de la relation strictement nécessaire entre les semigroupes fortement continus et la structure des espaces de Banach sous-jacents, pour l'étude de la théorie du contrôle. Particulièrement importantes sont les relations qui concernent l'adjoint d'un semigroupe fortement continu qui peut être factorisé par le biais d'un espace d'Asplund dans ce cas. L'adjoint du semigroupe est fortement continu sur $(0,\infty)$, ce qui est très important dans la théorie du contrôle. Dans le cas où le semigroupe en considération est compact, le système de contrôle associé ne peut jamais être exactement contrôlable en temps fini.

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1. Preface

In these notes we present the basic material of the workshop on Semigroup of Operators and Control Theory delivered by the first author in *CIMPA School on Orthogonal Family and Semigroups on Analysis and Probability* at University of Los Andes, Mérida, Venezuela from January 30 to February 11, 2006.

The course was based in the notes written by the authors on *Semigrupos Fuertemente Continuos y Algunas Aplicaciones*, which were written as a supporting material for a corresponding course for *XV Escuela Venezolana de Matemáticas* and *II Escuela Matemáticas de América Latina y el Caribe*, the text does not include the chapter III on differential equations of the original manuscript because it is out of the taught at CIMPA School.

The idea of this presentation is to describe some applications of the theory of semigroup in control theory. It does not pretend to be a survey on the subject. It only tries to illustrate the topic on which the authors lately have been interested in. For these reasons, these notes have been structured in the following three chapters.

Chapter I deals with some preliminary facts such as some generalities on Banach spaces and vector measures, Bochner integral and Radon Nikodym Property in Banach spaces. The material on Banach spaces is classical and can be found in several books on Functional Analysis quoted in our references, while the facts about vector measures can be found in Diestel-Uhl [**17**] and van Dulst [**18**].

Chapter II is devoted to the study of the adjoint semigroup for which a nice treatment can be found in van Neerven [**29**] and Nagel [**28**]. Our presentation is inspired by [**3**], thinking on some applications appearing in chapter III.

Chapter III deals with the structure of the range of the semigroup and the corresponding action on the continuity of the adjoint semigroup together with the application to control theory.

The results on measurable multifunctions and null controllability in reflexive Banach spaces come from [**1**], [**2**] and the general case from [**20**] and [**3**]. Many results regarding to exact and approximate controllability are taken from [**11**] and [**12**]; further developments and generalization can be found in [**6**] [**7**] [**8**] and [**20**]; in particular, proposition 2.6 is taking from [**6**], where it is proved in a more general setting. The bibliography does not pretend to be complete; we only quoted those papers used in the written of this work. No indication of the source of a result does not mean it appears the first time in this work.

2. Preliminaries

In this chapter we present some results from Functional Analysis which will be used in this work. Particulary, characterizations of surjective operators, the Bochner Integral and the Radon Nikodym Property.

3. Characterization of Surjective Operators

***Theorem 3.1* (Open Mapping Theorem).** — *Let X and Y be Banach spaces and $T \in L(X,Y)$ a surjective operator. Then there is $\alpha > 0$ such that*

$$T(B_X(0,1)) \supset B_Y(0,\alpha).$$

Theorem 3.2. — *Let X and Y be Banach spaces and consider $T \in L(X,Y)$ with $RangT = Y$. Then, there is $\alpha > 0$ such that for $S \in L(X,Y)$ with $\|T - S\| < \alpha \Longrightarrow RangS = Y$.*

Proof. Suppose $T \in L(X,Y)$ and $\mathrm{Rang}T = Y$. Then by the Open Mapping Theorem there exists $\alpha \in (0,1)$ such that

$$\alpha B_Y \subseteq T(\frac{1}{2}B_X) \subseteq T(B_X). \tag{3.1}$$

Assume that $S \in L(X,Y)$ with $\|T - S\| < \alpha$, and consider $y_0 \in \alpha B_Y$. Then, there exists $x_0 \in X$ such that $Tx_0 = y_0$, and putting $y_1 = Sx_0$ we obtain

$$\|y_1 - y_0\| = \|Tx_0 - Sx_0\| \leq \|T - S\| \leq \alpha.$$

Next, since $y_0 - y_1 - y_2 \in \frac{1}{2}B_Y \subset \frac{1}{2}T(\frac{1}{2^2}B_X)$, there exists $x_2 \in \frac{1}{2^2}B_X$ such that $Tx_2 = y_0 - y_1 - y_2$. Now, putting $y_3 = Sx_2$ we obtain that

$$\|y_0 - y_1 - y_2 - y_3\| = \|Tx_2 - Sx_2\| \leq \frac{\alpha}{2^2}.$$

In this way we construct a pair of sequences $(x_n)_{n=0}^{\infty} \subset X$ and $(y_n)_{n=0}^{\infty} \subset Y$ such that

$$y_n = Sx_{n-1} \quad \text{and} \quad Tx_n = y_0 - \sum_{i=1}^{n} y_i,$$

with $\|Tx_n\| \leq \frac{\alpha}{2^{n-1}}$ and $\|x_n\| \leq \frac{1}{2^n}\ n, \quad \in \mathbb{N}$.
Since X is a Banach space, the serie $\sum_{n=0}^{\infty} x_n$ converges to some $x \in X$, and

$$Sx = S(\sum_{i=0}^{\infty} x_i) = \sum_{i=0}^{\infty} S(x_i) = \sum_{i=1}^{\infty} y_i.$$

Since $\|y_0 - \sum_{i=1}^{\infty} x_i\| = \|Tx_n\| \to 0$, we see that $Sx = y_0$. Therefore S is surjective. ■

Theorem 3.3. — *Let X, Y, Z be Banach spaces and consider $T \in L(X,Z)$ and $S \in L(Y,Z)$. If $RangeT \subset RangeS$, then there exists $\gamma > 0$ such that*

$$\gamma \|S^* z^*\|_{Y^*} \geq \|T^* z^*\|_{X^*} \quad \forall\ z^* \in Z^*.$$

Proof. We first suppose that S is one to one. Then $S^{-1} : Range S \longrightarrow Y$ is well defined. Moreover $S^{-1} \circ T \in L(X,Y)$; hence its adjoint operator is bounded, indeed there exists $\gamma > 0$ such that

$$\|(S^{-1}T)^* y^*\|_{X^*} \leq \gamma \|y^*\|_{X^*} \quad \forall\, y^* \in Y^*.$$

On the other hand,

$$\begin{aligned} \langle (S^{-1} \circ T)^* y^*, x \rangle_{X^*,X} &= \langle (S^{-1} \circ T)^* S^* z^*, x \rangle_{X^*,X} \\ &= \langle S^* z^*, (S^{-1} \circ T) x \rangle_{Y^*,Y} \\ &= \langle z^*, S S^{-1} T x \rangle_{Z^*,Z} \\ &= \langle z^*, Tx \rangle_{Z^*,Z} = \langle T^* z^*, y \rangle \end{aligned}$$

From here we have that

$$(S^{-1}T)^* y^* = T^* z^*$$

which implies

$$\|T^* z^*\|_{X^*} \leq \gamma \|S^* z^*\|_{Y^*}.$$

For the general case, consider the quotient space $\tilde{Y} = Y/\ker S$ which is a Banach space with the norm

$$\|[y]\| = \inf_{S(\tilde{y})=0} \|y + \tilde{y}\|,$$

where $[y] \in \tilde{Y}$ denotes the equivalence class of y.
Now, we define

$$\tilde{S} : \tilde{Y} \longrightarrow Z, \quad \text{as } \tilde{S}([y]) = S(y).$$

$\tilde{S}$ is one to one and $Range T \subset Range \tilde{S}$. From the first part of the proof of this theorem we have:

$$\gamma \|\tilde{S}^* z^*\|_{\tilde{Y}^*} \geq \|T^* z^*\|_{X^*} \quad \forall\, z^* \in Z^*.$$

From the definition of the quotient map $\tilde{S}$ we have that

$$\|\tilde{S}^* z^*\|_{\tilde{Y}^*} = \|S^* z^*\|_{Y^*},$$

and the theorem is proved. ■

Theorem 3.4. — *If $T \in L(X,Y)$ and $S \in L(X,Z)$, where X, Y and Z are Banach spaces, and there is a constant $\gamma > 0$, such that*

$$\|Tx\|_Y \leq \gamma \|Sx\|_Z, \qquad \forall\, x \in X$$

then

$$Range(T^*) \subset Range(S^*).$$

Proof. Take $x^* \in Range T^*$, indeed $x^* = T^* y^*$ for some $y^* \in Y^*$. We want to find $z^* \in Z^*$ such that $x^* = S^* z^*$. This is equivalent to

$$\langle y^*, Tx \rangle_{Y^*,Y} = \langle z^*, Sx \rangle_{Z^*,Z} \quad \forall\, x \in X.$$

Let f be a function defined from Range(S) to $\mathbb{C}$ by

$$f Sx = \langle y^*, Tx \rangle_{Y^*,Y};$$

whence

$$\begin{aligned} |f(Sx)| &= |\langle y^*, Tx\rangle| \\ &\leq \gamma\|y^*\|\|Tx\|_Z \\ &\leq \gamma\|y^*\|\|Sx\|_Z; \end{aligned}$$

thus f is a bounded linear functional. So by Hahn-Banach theorem we can extend f to the whole space Z; indeed, there exist $z^* \in Z^*$ such that

$$f(Sx) = \langle z^*, Sx\rangle_{Z^*,Z} \quad \forall\, x \in X.$$

Consequently

$$\langle y^*, Tx\rangle_{Y^*,Y} = \langle z^*, Sx\rangle_{Z^*,Z} \quad \forall\, x \in X,$$

therefore

$$x^* = T^*y^* = S^*z^*.$$

■

Corollary 3.1. — *Let X, Y, Z be reflexive Banach spaces, $T \in L(X,Z)$, and $S \in L(Y,Z)$. Then the following are equivalent conditions:*

i) *$RangeT \subset RangeS$*
ii) *There is $\gamma > 0$ such that*

$$\gamma\|S^*y^*\| \geq \|x^*\|_{X^*} \qquad \forall\, x^* \in X^*.$$

Corollary 3.2. — *If X, Y are Banach spaces and $S \in L(X,Y)$, the following are equivalent conditions:*

i) *$RangeS = Y$*
ii) *There is $\gamma > 0$, such that*

$$\gamma\|S^*y^*\|_{Y^*} \geq \|T^*x^*\|_{X^*} \qquad \forall\, x^* \in X^*$$

Theorem 3.5. — *Let $T \in L(X,Y)$ and $S \in L(Z,Y)$ where X, Y and Z are Banach spaces. Then the following conditions are equivalent:*

i) $\ker(S^*) \subset \ker(T^*)$
ii) $\overline{RangeS} \supset \overline{RangeT}$

Proof. Suppose $\ker S^* \subset \ker T^*$ and $\overline{RangeS}$ does not contain $\overline{RangeT}$. Then there is $y \in \overline{RangeT}$ such that $y \notin \overline{RangeS}$. Hence there is $y^* \in Y^*$ with $y^* \neq 0$ such that $\langle y^*, y\rangle \neq 0$ and $y^*|_{RangeS} \equiv 0$. Therefore

$$\langle y^*, Sz\rangle_{Z^*,Z} = 0 \qquad \forall\, z \in Z.$$

$\Longrightarrow y^* \in \ker S^*$, and consequently $y^* \in \ker T^*$. Thus

$$\langle y^*, Tx\rangle = 0 \qquad \forall\, x \in X.$$

Since $y \in \overline{RangeT}$, there exists a sequence $(x_n) \subset X$ such that $y = \lim\limits_{n\to\infty} Tx_n$. Therefore

$$0 = \lim_{n\to 0}\langle T^*y^*, x_n\rangle = \lim_{n\to\infty}\langle y^*, Tx_n\rangle = \langle y^*, y\rangle$$