Séminaires & Congrès
25, 2012, p. 111–193

LIE ALGEBRAS, REPRESENTATIONS, AND ANALYTIC SEMIGROUPS THROUGH DUAL VECTOR FIELDS

by

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Abstract. — We want to present the basics of a new point of view in a variety of areas using the idea of *Dual Vector Fields.* These topics include operator calculus, representations of Lie algebras, analytic semigroups, and probability semigroups.

Résumé **(Algèbres de Lie, représentations, et semigroupes analytiques par champs de vecteurs duals)**

Nous voulons présenter les fondements d'un nouveau point de vue dans une multitude des domaines, en utilisant l'idée des *Champs de Vecteurs Duals.* Ces sujets comprennent le calcul opérationnel, les représentations des algèbres de Lie, semigroupes analytiques et semigroupes probabilistes.

LECTURE I

COHERENT STATE REPRESENTATIONS: OPERATORS AND DUALITY

Let us start with the basics of operators and duality with some examples relating to probability theory.

2010 Mathematics Subject Classification. — 17B66, 17B99, 33A65, 33A75, 33C80, 37C10, 46L53, 60E05.

Key words and phrases. — Lie algebras, adjoint group, Appell systems, orthogonal polynomials, representations of Lie groups, special functions, probability and algebraic structures, quadratic variance, Leibniz function, quantum probability, dual vector fields
algèbres de Lie, groupe adjoint, systèmes d'Appell, polynômes orthogonaux, représentations des groupes de Lie, fonctions spéciales, structures probabilistes et algébriques, variance quadratique, fonction de Leibniz, probabilités quantiques, champs de vecteurs duals.

1. Simple Fock spaces

We have a *vector space* $\mathcal{H}$ with a basis $\{\psi_n\}_{n\geq 0}$. Throughout, our scalars will be $\mathbf{C}$, the complex numbers, or alternatively, we restrict to $\mathbf{R}$, the real numbers.

The *Dirac notation* writes $\psi_n = |\,n\,\rangle$, called "ket", where the label n is the eigenvalue of an operator on $\mathcal{H}$. In this case, it is the *number operator*, $\mathcal{N}$, $\mathcal{N}\psi_n = n\,\psi_n$. In other words, $\mathcal{N}$ is diagonal in this basis with eigenvalues $\{0,1,2,\ldots\}$. In realizing these as functions, it is convenient to label them according to the number of underlying variables. For d variables, $\{\,x_1,\ldots,x_d\,\}$, we write the basis as $\psi_n = \psi_{n_1,\ldots,n_d} = |\,n_1,\ldots,n_d\,\rangle$, so that n denotes the corresponding multi-index $(n_1,\ldots,n_d)$, with number operators $\mathcal{N}_i\psi_n = n_i\,\psi_n$. Then $\mathcal{N} = \sum_i \mathcal{N}_i$ acts as $\mathcal{N}\psi_n = |n|\,\psi_n$, the total degree of ψ_n. The state $|\,0\,\rangle$ is called the *vacuum state*, is often denoted by Ω, and is mapped to the zero vector by all lowering operators.

1.1. Raising and lowering operators.

— For a single index, introduce *raising* and *lowering* operators, $\mathcal{R}$ and $\mathcal{V}$.

$$\mathcal{R}\,|\,n\,\rangle = |\,n+1\,\rangle, \qquad \mathcal{V}\,|\,n\,\rangle = n|\,n-1\,\rangle$$

Think of going from $x^n \to x^{n+1}$ by multiplying by x, and correspondingly from $x^n \to nx^{n-1}$ by differentiation. The specific operators analogous to differentiation are denoted by $\mathcal{V}$'s and referred to as *velocity operators* as "lowering operator" refers more generally to any operator lowering the degree. For d variables, we have

$$\mathcal{R}_i\,|\,n\,\rangle = |\,n+\mathrm{e}_i\,\rangle = |\,n_1,\ldots,n_i+1,\ldots,n_d\,\rangle, \qquad \mathcal{V}_i\,|\,n\,\rangle = n_i|\,n-\mathrm{e}_i\,\rangle$$

where e_i is a vector of 0's except for a 1 in the i^{th} spot.

1.2. Lie algebras.

— A *Lie algebra*, $\mathfrak{g}$, is an algebra where the multiplication, denoted by brackets $[a,b]$, satisfies $[a,a]=0$ and the *Jacobi identity*

$$[a,[b,c]] + [c,[a,b]] + [b,[c,a]] = 0$$

In our case, we will use the Lie product given by $[a,b] = ab - ba$, the *commutator* on an associative algebra.

A **representation** of $\mathfrak{g}$ is a realization of $\mathfrak{g}$ where the elements are given as linear maps on a vector space and the Lie product maps to the commutator. The action of a as a linear map on $\mathfrak{g}$ given by $b \to [a,b]$ is the *adjoint representation*, the mapping written as

$$(\operatorname{ad} a)(b) = [a,b]$$

Typically a Lie algebra is specified by prescribed commutation relations on a basis. Elements a and b *commute* if $[a,b]=0$. Commutation relations between commuting elements are not explicitly indicated.

Throughout, we will use $\{\,\xi_1,\xi_2,\ldots,\xi_d\,\}$ as the basis for a d-dimensional Lie algebra. Then the Lie algebra is determined by the linear maps

$$(\operatorname{ad}\xi_k)(\xi_j) = [\xi_k,\xi_j] = \sum_i c^i_{kj}\xi_i$$

The coefficients c^i_{kj} are called the *structure constants* of the Lie algebra. These determine matrices of the adjoint representation, which we denote by $\check{\xi}_k$,

$$(\check{\xi}_k)_{ij} = c^i_{kj}$$

The fact that this is a representation follows from the Jacobi identity.

We work mainly with operators acting on polynomials and by extension to holomorphic functions defined in some given neighborhood of 0, which we call *locally holomorphic functions.* Alternatively, we can use formal power series. We refer to these three classes of objects as "suitable functions".

The *Heisenberg-Weyl* algebra is given by the commutation rule

$$[\xi_3, \xi_1] = \xi_2$$

where it is implicit that ξ_2 is in the **center**, i.e., it commutes with ξ_1 and ξ_3. A matrix representation of the HW algebra is

$$\xi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that the adjoint representation is different:

$$\check{\xi}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \check{\xi}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \check{\xi}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

1.3. Representations of HW. — Now, notice that, for one variable, $\mathcal{R}$ and $\mathcal{V}$ acting on the vectors $|\, n\,\rangle$ satisfy $[\mathcal{V}, \mathcal{R}] = I$, where I is the identity operator, i.e.,

$$(\mathcal{V}\mathcal{R} - \mathcal{R}\mathcal{V})|\, n\,\rangle = (n + 1 - n)|\, n\,\rangle = |\, n\,\rangle$$

And I commutes with all operators. So this is a representation of the HW algebra.

Remark 1.1. — We will usually identify a multiple of the identity operator, say, cI, with the number c.

Let's use the realization of operators on polynomials as follows. We denote

X operator of multiplication by x, $\qquad D$ differentiation with respect to x

The basis is $|\, n\,\rangle = x^n$, with $|\, 0\,\rangle = 1$. For polynomials in d variables, we have correspondingly X_i as multiplication by x_i and D_i partial differentiation with respect to x_i. Note the commutation relations

$$[D_j, X_i] = \delta_{ij}\, I$$

which prescribe the d-dimensional HW algebra. Any family of operators $\{\, \mathcal{R}_i, \mathcal{V}_j\,\}$ satisfying analogous commutation relations are called *boson operators* in quantum probability.

Note that any Lie algebra may be realized using first-order differential operators, **vector fields**, by the mapping,

$$\xi_i \leftrightarrow X_\lambda c_{i\mu}^\lambda D_\mu$$

called the Jordan map.

Notation 1.2. — Our summation convention is: *Greek indices are always summed.*

When we have specific realizations of $\mathcal{R}$'s and $\mathcal{V}$'s acting on polynomials or spaces of functions, we denote the corresponding operators by R's and V's.

1.4. Examples in probability theory. — Interesting examples are available from probability theory. We look at the *moment polynomials* arising from a distribution and we look at certain families of orthogonal polynomials for some probability distributions.

1.4.1. *Gaussian*. — Let $p_t(dx) = \dfrac{e^{-x^2/(2t)}}{\sqrt{2\pi t}}\,dx$ be the Gaussian density with mean zero, variance $t > 0$. Defining

$$h_n(x) = \int_{-\infty}^{\infty} (x+y)^n \; p_t(dy) = \int_{-\infty}^{\infty} (x + y\sqrt{t})^n p_1(dy) \tag{1}$$

we write this, using angle brackets to denote expected value, as

$$\langle (x + X_t)^n \rangle$$

where X_t is the corresponding Gaussian variable.

One sees that $V = D$, i.e., $Dh_n = n\,h_{n-1}$. The raising operator, R, is no longer X, but, in fact, is $R = X + tD$. This can be written as a **recurrence formula**. Another way to think of it as a realization of X in terms of R and V. From $X = R - tD = R - tV$ we have

$$x\,h_n = h_{n+1} - tn\,h_{n-1}$$

It turns out that a family of *Hermite polynomials* is orthogonal with respect to this distribution. They are given by

$$H_n(x) = \int_{-\infty}^{\infty} (x + iy)^n \; p_t(dy)$$

where $i = \sqrt{-1}$. From the second formulation in equation (1), we see that one has replaced $t \to -t$. Thus,

$$R = X - tD, \qquad V = D$$

for the Hermite polynomials. The recurrence is thus

$$x\,\psi_n = h_{n+1} + tn\,h_{n-1}$$

which is the **three-term recurrence** a family of orthogonal polynomials must satisfy.

Notice that $R^* = tV$, the operator adjoint to R with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(y)g(y)\,p_t(dy)$$

on polynomials or smooth functions with derivatives in $L^2(\mathbf{R})$ of the corresponding Gaussian measure.

1.4.2. *Poisson.* — Now consider the Poisson distribution, with

$$p_t(x) = e^{-t}\,\frac{t^x}{x!}$$

for integer $x \geq 0$. The *Poisson-Charlier polynomials* are orthogonal with respect to this Poisson distribution. They have generating function

$$G(v) = G(v; x, t) = (1+v)^x\, e^{-vt} = \sum_{n\geq 0} \frac{v^n}{n!}\, P_n(x, t)$$

Verifying that $\langle G(v)G(w)\rangle$ is a function of vw alone shows that the polynomials P_n are indeed orthogonal. We have the *difference operator* expressed in terms of D by

$$(e^D - 1)f(x) = f(x+1) - f(x)$$

on polynomials (in general, suitable functions). Notice the duality "multiplication by v" and the lowering operator $VP_n = nP_{n-1}$. Acting on G, we see that $V = e^D - 1$. The raising operator R is dual to differentiation with respect to v. In other words, the operators V, R are given by transferring the action of the HW representation "multiplication by v, differentiation with respect to v" via the generating function G to the sequence $\{\, P_n \,\}$. We must express the result of differentiating with respect to v in terms of X and D. Noting that

$$\frac{1}{1+V} = e^{-D}$$

we find the HW representation

$$R = Xe^{-D} - tI, \qquad V = e^D - I$$

Solving, we find

$$X = (R+t)(1+V) = t + R + RV + tV$$

Note that RV is the number operator. Thus the recurrence formula

$$xP_n = P_{n+1} + (n+t)P_n + ntP_{n-1}$$

1.4.3. *Analytic HW realizations.* — To see why we expect that $[V, R] = I$ from the above formulas, we first note that for any polynomial $f(x)$, inductively it follows that $[V, f(R)] = f'(R)$ acting on kets. Dually, $[f(V), R] = f'(V)$. So the analogous formulas hold for all boson operators. These extend to suitable functions f. In particular, if $V(z)$ denotes a locally holomorphic function, such that $V(0) = 0$, $V'(0) \neq 0$, we define *canonical boson operators* associated to V by

$$R = XW(D), \qquad V = V(D)$$

where $W(D) = V'(D)^{-1}$, a notation to be used consistently throughout. The vacuum for the representation is the function equal to 1.