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ANALYSIS OF ORNSTEIN–UHLENBECK AND LAGUERRE STOCHASTIC PROCESSES

by

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Abstract. — The objective of these lectures is to present Ornstein–Uhlenbeck and related stochastic processes to a wide mathematical audience with a modest preparation in stochastic analysis.

The aim of the first part of the lectures (Chapter 1) is to discuss the Ornstein–Uhlenbeck and the Squared Radial Ornstein–Uhlenbeck stochastic **diffusion** processes, whose infinitesimal generators are, respectively, the Ornstein–Uhlenbeck operator and the Laguerre operator.

In the second chapter of these lectures the Ornstein–Uhlenbeck processes governed by α–**stable** rotationally invariant processes are studied. This corresponds to replacing the Laplacian Δ by the fractionnary Laplacian $-(-\Delta)^{\alpha/2}$ in the Ornstein–Uhlenbeck generator $L = \frac{1}{2}\Delta - x \cdot \nabla$. More general drift terms $b(x) \cdot \nabla$ are also considered at the end of Chapter 2.

Résumé **(Analyse des Processus Stochastiques d'Ornstein–Uhlenbeck et de Laguerre)**

L'objectif de ce cours est de présenter le processus stochastique d'Orns–tein–Uhlenbeck et les processus y liés, à un large public mathématique avec une préparation modeste en analyse stochastique.

Le but de la première partie du cours (Chapitre 1) est de donner une présentation des diffusions d'Ornstein–Uhlenbeck et du processus d'Ornstein–Uhlenbeck radial au carré. Leurs générateurs infinitésimaux sont, respectivement, l'opérator d'Ornstein–Uhlenbeck et l'opérateur de Laguerre.

Dans le deuxième chapitre de ce cours nous étudions les processus d'Ornstein–Uhlenbeck dirigés par les processus α–**stables**, invariants par rotations. Ceci correspond à remplacer le laplacien Δ par le laplacien fractionnaire $-(-\Delta)^{\alpha/2}$ dans le générateur d'Ornstein–Uhlenbeck $L = \frac{1}{2}\Delta - x \cdot \nabla$. Les termes de drift plus généraux $b(x) \cdot \nabla$ sont également considerés à la fin du chapitre 2.

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1. ORNSTEIN–UHLENBECK AND LAGUERRE DIFFUSIONS

1.1. Introduction. — The Ornstein–Uhlenbeck stochastic diffusion process is very well presented in the stochastic literature. It was introduced in [**Ornstein–Uhlenbeck**] already in 1930. For this reason, in these lectures we are not going to deal much with this classical diffusion, but only present some of its main features that are not always well realised.

1.2. Physical motivation of the Ornstein–Uhlenbeck diffusion. — The main reference for this section is [**Breiman**], p.347–350.

A stochastic process $\{Y_t : t \geq 0\}$ has **independent increments** if for all $t_1 < t_2 < \cdots < t_n$ the random variables $Y_{t_1} - Y_{t_0}$, $Y_{t_2} - Y_{t_1}$ are independent. The increments are further said to be **stationary** if, for any $t > s$ and $h > 0$, the distribution of $Y_{t+h} - Y_{s+h}$ is the same as the distribution of $Y_t - Y_s$.

The **Brownian motion(Wiener process)** was constructed as a model for a **microscopic particle in liquid suspension**. Recall that a stochastic process $\{B_t\}_{t\geq 0}$ is a Brownian motion if it has stationary independent increments (is a Lévy process), if B_t is normally $N(0,t)$–distributed and if $B_0 = 0$.

It follows immediately that that the process $\{B_t\}_{t\geq 0}$ is:

– Gaussian (a stochastic process Y_t is Gaussian if for all $t_1 < t_2 < \cdots < t_n$ the vector $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})$ is multivariate normally distributed)

– Markovian (i.e. for all $t_1 < t_2 < \cdots < t_n$ we have $P(Y_{t_n} \leq y |\, Y_{t_1}, Y_{t_2}, \ldots, Y_{t_{n-1}}) = P(Y_{t_n} \leq y | Y_{t_{n-1}})$

One may not require the normality of B_t but impose a physical **continuity of trajectories** condition that implies the normality of B_t.

An outstanding nonreality of the Brownian motion model is the assumption that increments in displacement are independent. We ignore in this way the effects of the velocity of the particle at the beginning of the incremental period. We can do better in the following way.

Let $V(t)$ be the **velocity** of a particle of **mass** m suspended in liquid. Set

$$\Delta V = V(t + \Delta t) - V(t).$$

Then $m\Delta V$ is the change of **momentum** of the particle during time Δt.

Let $M(t)$ be the momentum of a particle at time t. We consider ΔM, the momentum transfer due to molecular bombardment of the particle during time Δt. We should take into account a friction effect. Let $\beta > 0$ and $-\beta V$ the viscous resistance force. Consequently,

$$-\beta V \Delta t$$

is the loss in momentum due to viscous forces during Δt. Thus we have the basic equation

$$m\Delta V = -\beta V\Delta t + \Delta M. \tag{1}$$

Assume that

(i) M has independent increments

(ii) the distribution of ΔM depends only on Δt

(iii) $M(t)$ is continuous in t.

The third assumption of **continuity** of momentum may be questionnable(imagine a hard billiard–ball model of molecules). Such a situation will be considered in the next chapter (stable Ornstein–Uhlenbeck processes).

But (i),(ii) and (iii) characterize $M(t)$ as a Brownian motion, possibly with a constant drift μt, due to a constant force field and useful, for example, in accounting for a gravity field. However we will assume no constant force field exists, and set $\mathbb{E}M(t) = 0$ and $VarM(t) = \mathbb{E}M^2(t) = \sigma^2 t$. Hence

$$M(t) = \sigma B(t)$$

where $B(t)$ is normalized Brownian motion. Equation (1) becomes

$$m\Delta V = -\beta V\Delta t + \sigma\Delta B. \tag{2}$$

It is natural to divide by Δt and let $\Delta t \to 0$. We produce in this way the celebrated **Langevin equation**, the oldest example of a Stochastic Differential Equation

$$m\frac{dV}{dt} = -\beta V + \sigma\frac{dB}{dt}. \tag{3}$$

There is a difficulty here: we know that the trajectories of the Brownian motion are nowhere differentiable, so (3) makes no sense in any orthodox deterministic way. There is however a possibility of avoiding of stochastic Itô differential calculus. We reformulate (3) as an integral equation. Write (3) as

$$\frac{d}{dt}(e^{\alpha t}V(t)) = \gamma e^{\alpha t}\frac{dB(t)}{dt}$$

with $\alpha = \beta/m$ and $\gamma = \sigma/m$. Assume $V(0) = 0$ and integrate from 0 to t to get

$$e^{\alpha t}V(t) = \gamma\int_0^t e^{\alpha s}dB(s).$$

Do an integration by parts on the integral,

$$e^{\alpha t}V(t) = \gamma e^{\alpha t}B(t) - \gamma\alpha\int_0^t B(s)e^{\alpha s}ds.$$

Now the integral appearing is for each ω just the integral of a continuous function and makes sense.

Actually it is easy to see that the integral

$$\int_0^t e^{\alpha s}dB(s)$$

converges as a Riemann–Stieltjes integral. The approximating sums satisfy

$$\sum_{k=0}^{n-1} e^{\alpha t_k}(B(t_{k+1}) - B(t_k)) = e^{\alpha t}B(t) - \sum_{k=0}^{n-1}(e^{\alpha t_{k+1}} - e^{\alpha t_k})B(t_{k+1})$$

for any partition $0 = t_0 < \cdots < t_n = t$ of $[0, t]$. The last sum is the Riemann–Stieltjes sum for the integral

$$\int_0^t B(s)d(e^{\alpha s}).$$

For every sample path, the sum converges to this integral, if the diameter of the partition $0 = t_0 < \cdots < t_n = t$ goes to 0.

Another, more sophisticated but also more systematic argument is to use the fact that the Riemann–Stieltjes integral $\int_a^b f(s)dg(s)$ is well defined when f has bounded q–variation and g has bounded p–variation with $\frac{1}{p} + \frac{1}{q} > 1$. We also know that the trajectories of the Brownian motion B_t have bounded p-variation for any $p > 2$.

We have justified the following definition

Definition 1.1. — The Ornstein–Uhlenbeck process $V(t)$ starting at 0 is defined as

$$V(t) = \gamma \int_0^t e^{-\alpha(t-s)}dB(s),$$

where the integral is in the Riemann–Stieltjes pathwise sense.

The Ornstein–Uhlenbeck process modelizes the **velocity of a particle in liquid viscous suspension**. We show here some of its main properties.

Proposition 1.2. — *$V(t)$ is a Gaussian process with $\mathbb{E}V(t) = 0$ and covariance*

$$\mathbb{E}V(s)V(t) = \rho(e^{-\alpha|t-s|} - e^{-\alpha(s+t)}).$$

where $\rho = \gamma^2/2\alpha$.

Proof. — The process $V(t)$ is Gaussian because it is the limit, almost everywhere, of the sums

$$\sum_{k=0}^{n-1} e^{\alpha t_k}(B(t_{k+1}) - B(t_k))$$

where $\Delta_k B = B(t_{k+1}) - B(t_k)$ are independent, normally distributed random variables. To get the covariance, suppose $s > t$ and put

$$0 = t_0 < \cdots < t_n = t < t_{n+1} < \cdots t_m = s.$$

Then

$$V(s) = \lim \gamma \sum_{k=0}^{m} e^{-\alpha(s-t_k)}\Delta_k B$$

$$V(t) = \lim \gamma \sum_{k=0}^{n} e^{-\alpha(s-t_k)}\Delta_k B.$$

We compute the covariance of the approximating sums. Using

$$\mathbb{E}\Delta_k B \Delta_j B = (t_{k+1} - t_k)\delta_{kj}$$

we get

$$\mathbb{E}V(s)V(t) = \gamma^2 \lim \sum_{k=0}^{n} e^{-\alpha(s+t)+2\alpha t_k}(t_{k+1} - t_k) = \gamma^2 e^{-\alpha(s+t)} \int_0^t e^{2\alpha u} du.$$

□

Corollary 1.3. — $\lim_{t\to\infty} V(t) = N(0,\rho)$ *in law.*

Proof. — This follows from the normality of $V(t)$ and the fact that $\mathbb{E}V(t) = 0$ and $VarV(t) = \rho(1 - e^{-\alpha 2t}) \to \rho$. □

Stationarity. What if we start the Ornstein–Uhlenbeck process not at 0 but with the limiting distribution $N(0,\rho)$? The integration by parts in the Langevin equation gives now

$$e^{\alpha t}\tilde{V}(t) - \tilde{V}(0) = \gamma \int_0^t e^{\alpha s} dB(s).$$

so in general we have

$$\tilde{V}(t) = e^{-\alpha t}\tilde{V}(0) + \gamma \int_0^t e^{-\alpha(t-s)} dB(s) = e^{-\alpha t}\tilde{V}(0) + V(t)$$

where $V(t) = \gamma \int_0^t e^{-\alpha(t-s)} dB(s)$ is the Ornstein–Uhlenbeck process starting at 0.

Definition 1.4. — Let $V(t)$ be an Ornstein–Uhlenbeck process starting at 0.
Let $\tilde{V}(0)$ be independent of $\sigma(V(t), t \geq 0)$ and with normal $N(0,\rho)$ law. Define the Ornstein–Uhlenbeck process $Y(t)$ starting from $\tilde{V}(0)$ by

$$Y(t) = e^{-\alpha t}\tilde{V}(0) + V(t)$$

It is a solution of the Langevin equation with the original distribution $V(0) = \tilde{V}(0)$.

A stochastic process $\{Y_t : t \geq 0\}$ is called **stationary** if for all $t_1 < t_2 < \cdots < t_n$ and $h > 0$, the random vectors $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})$ and $(Y_{t_1+h}, Y_{t_2+h}, \ldots, Y_{t_n+h})$ are identically distributed; that is, time shifts leave joint probabilities(thus, by Kolmogorov theorem, the law of the process) unchanged.

Proposition 1.5. — *The process $Y(t)$ is a stationary Gaussian process with covariance $Cov(Y(s), Y(t)) = \rho e^{-\alpha|t-s|}$.*

Proof. — We compute the covariance of $Y(t)$ in the same way as in the proof of Proposition 1.2. The stationarity of $Y(t)$ may be shown by a direct computation or by the fact that a Gaussian process with zero means is stationary iff its covariance function $\Gamma(s,t) = \phi(|s-t|)$. □