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HERMITE AND LAGUERRE SEMIGROUPS SOME RECENT DEVELOPMENTS

by

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Abstract. — We introduce and study some properties of Hermite, special Hermite and Laguerre semigroups. The images of L^2 under these semigroups are shown to be certain weighted Bergman spaces of entire functions. The reader is expected to have some basic knowledge of Fourier Analysis but otherwise this notes is self-contained.

***Résumé* (Semigroupes d'Hermite et de Laguerre Quelques résultats récents).** — Nous introduisons et nous étudions quelques propriétés des semigroupes d'Hermite, des semigroupes spéciaux d'Hermite et des semigroupes de Laguerre. Nous montrons que les images de l'espace L^2 par ces semigroupes sont certains espaces de Bergman des fonctions entières, avec poids. Le lecteur devrait avoir les connaissances basiques de l'analyse de Fourier, sinon le cours est auto–suffisant.

1. Introduction

It is no exaggeration to say that Hermite functions are ubiquitous in Mathematics. They appear in such diverse fields as harmonic analysis, differential equations, mathematical physics and probability theory. They are eigenfunctions of the simple harmonic oscillator and hence play an important role in quantum mechanics. They are also eigenfunctions of the Fourier transform, a fact exploited by Norbert Wiener in his treatment of the Fourier transform. Hermite functions can be expressed as Laguerre functions of type $\frac{1}{2}$ and $-\frac{1}{2}$. In this sense Laguerre functions are generalisations of the Hermite functions. But there is a deeper relation between these two families of functions which arises in connection with analysis on the Heisenberg group $\mathbb{H}^n$.

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The modern theory of Hermite and Laguerre expansions makes use of this connection. Heisenberg group $\mathbb{H}^n$ is the most well known example from the realm of nilpotent Lie groups. No detailed harmonic analysis can be done on $\mathbb{H}^n$ without using Hermite functions. The most beautiful relation between Hermite and Laguerre functions is experessed by the formula $W(\varphi_k^{n-1}) = (2\pi)^n P_k$ where φ_k^{n-1} are Laguerre functions of type $(n-1)$, P_k are the projections associated to the Hermite operator and W is the Weyl transform related to the Schrodinger representation π_1 of $\mathbb{H}^n$.

Our aim in these lectures is to introduce the Hermite and Laguerre semigroups via the Heisenberg group. Both semigroups are related to the semigroup generated by the sublaplacian on $\mathbb{H}^n$. The group Fourier transform on $\mathbb{H}^n$ takes this latter semigroup into the Hermite semigroup whereas the Fourier decomposition in the central variable in $\mathbb{H}^n$ leads to the so called special Hermite semigroup. This last semigroup, generated by the special Hermite operator, encompasses all Laguerre semigroups of integer type.

After introducing the Hermite, special Hermite and Laguerre semigroups we proceed to the description of the image of L^2 under these semigroups. It is a classical result of Bargmann and Fock that the image of $L^2(\mathbb{R}^n)$ under the Gauss-Weierstrass semigroup can be defined as a weighted Bergman space of entire functions. This space was associated to the realisation of the creation and annihilation operators for Bosons in quantum physics. Similar results are known from the works of Hall [5] and Stenzel [10] for the semigroups generated by the Laplace-Beltrami operator on compact symmetric spaces. Recently it has been shown that the situation is quite different in the case of Heisenberg groups (see Krötz-Thangavelu-Xu [8]) and noncompact symmetric spaces (see Krötz-Olafsson-Stanton [9]).

The plan of the notes is as follows. We introduce Hermite and Laguerre semigroups in Section 2. In the process we introduce Segal-Bargmann transform on $\mathbb{R}^n$ and study the Fock-Bergman space associated to the standard Laplacian on $\mathbb{R}^n$. We also consider the Bessel semigroup and the associated Bergman spaces. In Section 3 we study the Hermite-Bergman and twisted Bergman spaces. The results for Laguerre semigroups are deduced from the corresponding results for Hermite and special Hermite semigroups.

2. Hermite and Laguerre semigroups

2.1. Hermite functions and Bargmann transform. — We begin with the definition of Hermite polynomials $H_k(x)$ where k is a nonnegative integer and $x \in \mathbb{R}$. These are defined by

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k}(e^{-x^2}).$$

It is then easy to see that the functions $\tilde{h_k}(x) = H_k(x) e^{-\frac{1}{2}x^2}$ are eigenfunctions of the Hermite operator $H = -\frac{d^2}{dx^2} + x^2$. More precisely,

$$(-\frac{d^2}{dx^2} + x^2)\tilde{h_k}(x) = (2k+1)\tilde{h_k}(x).$$

From this one can easily conclude that $\{\tilde{h_k} : k = 0, 1, 2, ...\}$ forms an orthogonal system in the Hilbert space $L^2(\mathbb{R})$. The operator H can be factorised as $H = \frac{1}{2}(AA^* + A^*A)$ where $A = \frac{d}{dx} + x$ and $A^* = -\frac{d}{dx} + x$ is its formal adjoint. The operators A and A^* are called the annihilation and creation operators respectively. Another easy calculation shows that

$$A^*\tilde{h_k}(x) = \tilde{h}_{k+1}(x), A\tilde{h_k}(x) = 2kh\tilde{_{k-1}}(x).$$

This shows that $\tilde{h_k}(x) = A^{*k}(e^{-\frac{1}{2}x^2})$. Using this and the relation $A^*A = H - 1$ along with an induction argument we can show that

$$\int_{\mathbb{R}} (\tilde{h_k}(x))^2 dx = 2^k k! \pi^{\frac{1}{2}}.$$

Therefore, we conclude that the functions

$$h_k(x) = (2^k k! \pi^{\frac{1}{2}})^{-\frac{1}{2}} \tilde{h_k}(x)$$

form an orthonormal family of functions in $L^2(\mathbb{R})$. We can actually show that they form an orthonormal basis (see below) which is called the Hermite basis in the literature.

From the definition of the Hermite polynomials it follows that H_k are given by the generating function

$$\sum_{k=0}^{\infty} \frac{H_k(x)}{k!} w^k = e^{2xw - w^2}$$

for any $w \in \mathbb{C}$. This can be checked by Taylor expanding the right hand side about $w = 0$ and using the definition of H_k. Defining $\zeta_k(w) = (2^k k! \pi^{\frac{1}{2}})^{-\frac{1}{2}} w^k$ we can rewrite the above as

$$\sum_{k=0}^{\infty} h_k(x)\zeta_k(w) = \pi^{-\frac{1}{2}} e^{-\frac{1}{2}(x-w)^2} e^{\frac{1}{4}w^2}. \tag{2.1}$$

The series converges uniformly over compact subsets of $\mathbb{C}$. From the above we can easily deduce

Theorem 2.1. — *The family* $\{h_k : k = 0, 1, 2, ...\}$ *is an orthonormal basis for* $L^2(\mathbb{R})$.

Proof. — As $\{h_k : k = 0, 1, 2, ...\}$ is an orthonormal system Bessel's inequality shows that

$$\sum_{k=0}^{\infty} |(f, h_k)|^2 \leq \|f\|_2^2.$$

Therefore, the series $\sum_{k=0}^{\infty} (f, h_k)\zeta_k(w)$ converges absolutely and equals

$$Bf(w) = \pi^{-\frac{1}{2}} e^{\frac{1}{4}w^2} \int_{\mathbb{R}} f(x) e^{-\frac{1}{2}(x-w)^2} dx.$$

If now $f \in L^2(\mathbb{R})$ is orthogonal to all h_k then the integral defining Bf will be zero for all w. This means convolution of f with the Gaussian $h_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}$ is identically zero. Taking Fourier transform we get $\hat{f} = 0$ and consequently $f = 0$. This proves the theorem. □

The operator B is called the Bargmann transform in the literature and has interesting properties. It takes functions $f \in L^2(\mathbb{R})$ into entire fuctions $Bf(w)$ on $\mathbb{C}$. These are not merely entire functions but also square integrable with respect to the Gaussian measure $d\mu(w) = (4\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}|w|^2} dw$. An easy calculation using polar coordinates show that the fuctions ζ_k form an orthonormal system in $L^2(\mathbb{C}, d\mu(w))$. Let $\mathcal{F}$ be the subspace of $L^2(\mathbb{C}, d\mu(w))$ consisting of entire functions. Then the equation

$$\sum_{k=0}^{\infty} (f, h_k)\zeta_k(w) = Bf(w)$$

shows that

$$\int_{\mathbb{C}} |Bf(w)|^2 d\mu(w) = \int_{\mathbb{R}} |f(x)|^2 dx.$$

This leads to the interesting result

Theorem 2.2. — *The Bargmann transform B is an isometric isomorphism between $L^2(\mathbb{R})$ and $\mathcal{F}$.*

Proof. — We only need to check that B is onto but this will follow once we observe that B takes the Hermite basis into $\{\zeta_k : k = 0, 1, 2, ...\}$ and this family is an orthonormal basis for $\mathcal{F}$. This last claim is justified since the expansion of a function F from $\mathcal{F}$ in terms of ζ_k is nothing but its Taylor expansion. □

Before proceeding further let us introduce multi-dimensional Hermite functions. For each multi-index $\alpha \in \mathbb{N}^n$ and $x \in \mathbb{R}^n$ we define

$$\Phi_\alpha(x) = h_{\alpha_1}(x_1).....h_{\alpha_n}(x_n).$$

Then it is clear that $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. We define the functions ζ_α and the space $\mathcal{F}(\mathbb{C}^n)$ in a similar way. More generally, for any $t > 0$ we define Fock spaces $\mathcal{F}_t(\mathbb{C}^n)$ as follows.

Definition 2.3. — *$\mathcal{F}_t(\mathbb{C}^n)$ is the space of all entire functions on $\mathbb{C}^n$ for which*

$$\|F\|^2_{\mathcal{F}_t} = \int_{\mathbb{C}^n} |F(w)|^2 e^{-t|w|^2} dw < \infty.$$

Note that $\mathcal{F}(\mathbb{C}^n) = \mathcal{F}_{\frac{1}{2}}(\mathbb{C}^n)$. The functions ζ_α form an orthonormal basis for $\mathcal{F}(\mathbb{C}^n)$ and the Bargmann transform is an isometric isomorphism between $L^2(\mathbb{R}^n)$ and $\mathcal{F}(\mathbb{C}^n)$.

We conclude this subsection with another useful formula known as Mehler's formula for the Hermite functions.

Proposition 2.4. — *For all $w \in \mathbb{C}$ with $|w| < 1$ we have*

$$\sum_{\alpha \in \mathbb{N}^n} \Phi_\alpha(x)\Phi_\alpha(y) w^{|\alpha|} = \pi^{-\frac{n}{2}}(1 - w^2)^{-\frac{n}{2}} e^{-\frac{1}{2}\frac{1+w^2}{1-w^2}(x^2+y^2) + \frac{2w}{1-w^2} x\cdot y}.$$

Proof. — It is enough to prove the formula in one dimension. As the Bargmann transform B takes the Hermite basis into the basis ζ_k it is unitary and hence its inverse is given by the adjoint B^*. Hence

$$\sum_{k=0}^{\infty} h_k(x)h_k(y)w^k = \sum_{k=0}^{\infty} h_k(x)B^*\zeta_k(y)w^k$$

and so in view of (2.1) we need to caculate $B^*F_w(y)$ when

$$F_w(z) = \pi^{-\frac{1}{2}}e^{-\frac{1}{2}(x-wz)^2}e^{\frac{1}{4}w^2z^2}.$$

Since

$$(B^*F, f) = (4\pi)^{-\frac{1}{2}}\int_{\mathbb{C}} F(z)Bf(z)e^{-\frac{1}{2}|z|^2}dz$$

an easy calculation shows that

$$B^*F(y) = (4\pi)^{-\frac{1}{2}}\int_{\mathbb{C}} F(z)e^{-\frac{1}{2}(y-z)^2}e^{\frac{1}{4}z^2}e^{-\frac{1}{2}|z|^2}dz.$$

Taking $F(z) = F_w(z) = \pi^{-\frac{1}{2}}e^{-\frac{1}{2}(x-wz)^2}e^{\frac{1}{4}w^2z^2}$ in this formula and evaluating the Gaussian Fourier transform we complete the proof. □

2.2. Gauss-Weierstrass kernel and Bergman spaces. — The space $\mathcal{F}(\mathbb{C}^n)$ is known as the Fock space in the literature. As a motivation for what we plan to do with the Hermite and Laguerre semigroups let us look at $\mathcal{F}(\mathbb{C}^n)$ more closely. Consider the Gauss-Weierstrass kernel or simply the heat kernel associated to the standard Laplacian Δ on $\mathbb{R}^n$ defined by

$$g_t(x) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{x^2}{4t}}.$$

(Here and later in these notes we will be writing x^2 in place of $|x|^2 = \sum_{j=1}^n x_j^2$. By the same convention for $z \in \mathbb{C}^n$ we let z^2 stand for $\sum_{j=1}^n z_j^2$.) The name heat kernel is justified since the function

$$G_tf(x) = f * g_t(x) = \int_{\mathbb{R}^n} f(y)g_t(x-y)dy$$

satisfies the heat equation for the Laplacian with initial condition f. Here f can be from any of the L^p spaces over $\mathbb{R}^n$.

For $w \in \mathbb{C}^n$ the heat kernel $g_t(w-y)$ makes sense as an entire function and so is $G_tf(x)$. That is to say the function $G_tf(x)$ extends to $\mathbb{C}^n$ as an entire function of $w = u+iv$. Note that for any $t > 0$ we have the relation

$$G_tf(w) = 2^{-\frac{n}{2}}Bf_t((2t)^{-\frac{1}{2}}w)e^{-\frac{1}{8t}w^2}$$

where $f_t(x) = f((2t)^{\frac{1}{2}}x)$ for all $f \in L^2(\mathbb{R}^n)$. The isometry between $L^2(\mathbb{R}^n)$ and $\mathcal{F}(\mathbb{C}^n)$ takes the form

$$\int_{\mathbb{C}^n} |G_{\frac{1}{2}}f(u+iv)|^2e^{-v^2}dudv = 2^{-n}\int_{\mathbb{R}^n} |f(x)|^2dx.$$