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SEMIGROUPS OF OPERATORS FOR CLASSICAL ORTHOGONAL POLYNOMIALS AND FUNCTIONAL INEQUALITIES

by

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Abstract. — In these notes we study the general theory of Markov semigroups establishing their basic properties and several other results. In particular we will study semigroups associated with classical orthogonal polynomials (Ornstein-Uhlenbeck semigroup, Laguerre semigroup and Jacobi semigroup). We will study in detail the hypercontractivity property of the Ornstein-Uhlenbeck semigroup. In order to do that we will prove that the Ornstein-Uhlenbeck operator satisfies a logarithmic Sobolev inequality which is an equivalent condition as it was proved by Leonard Gross. Then we will study functional inequalities, which relate the $L^p(\mu)$ norm of a function to the $L^q(\mu)$ norm of its (weak) gradient (Sobolev inequalities, logarithmic Sobolev inequalities, and spectral gap inequalities). Finally, we will also consider curvature-dimension inequalities.

Résumé **(Semigroupes des Opérateurs pour les Polynômes Orthogonaux Classiques et les Inégalités Fonctionnelles)**

Ce cours est consacré à la théorie des semigroupes de Markov. Il présente leurs propriétés fondamentales et quelques autres résultats. En particulier nous étudions les semigroupes associés aux familles classiques des polynômes orthogonaux (semigroupe d'Ornstein-Uhlenbeck, de Laguerre et de Jacobi) Nous allons étudier en détail la propriété d'hypercontractivité du semigroupe d'Ornstein-Uhlenbeck. Pour cela, nous montrons que l'opérateur d'Ornstein-Uhlenbeck vérifie une inégalité logarithmique de Sobolev, ce qui est équivalent à l'hypercontractivité, comme dé montré par Leonard Gross. Ensuite nous étudions les inégalités fonctionnelles, qui relient la norme $L^p(\mu)$ d'une fonction à la norme $L^q(\mu)$ de son (faible) gradient (inégalités de Sobolev, inégalités de Sobolev logarithmiques et les inégalités du trou spectral). Finalement, nous considérons aussi les inégalités courbure–dimension.

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1. Introduction

The theory of operator semigroups is a point of confluence of several areas of Mathematics. These include functional analysis, harmonic analysis, the theory of orthogonal polynomials, differential equations, the theory of probability and control theory, among others.

In these notes we study the analytical theory of operator semigroups associated with classical orthogonal polynomials: the Ornstein-Uhlenbeck semigroup, the Laguerre semigroup and the Jacobi semigroup. We will refer to these semigroups as *classical semigroups*, although it would be more accurate to call them "semigroups generated by classical orthogonal polynomials."

The present notes are strongly influenced by the work of Dominique Bakry ([**10**], [**11**], [**12**] and [**14**]). In particular by his notes for the CIMPA School at the Tata Institute for the Mathematical Science, Mumbay, India [**12**]. For that reason we are going to develop semigroup theory from the point of view of probability, focusing mainly on Markov semigroups in a probability space $(E, \mathcal{B}, \mu)$. Then we study functional inequalities such as the Sobolev inequalities, logarithmic Sobolev inequalities, spectral gap inequalities, and curvature-dimension inequalities. Those inequalities will allow us to study a very important property for these semigroups, the hypercontractivity property. As a consequence of this property we establish the well known multiplier theorem of P.A. Meyer, not only for expansions in Hermite polynomials, but also for Laguerre and Jacobi polynomials.

The notes are divided into four sections. Section 1 serves as an introduction. In section 2 we study the theory of Markov semigroups, establishing their basic properties and several other results. In particular, we discuss general properties of Markov semigroups associated with a family of orthogonal polynomials. In section 3 we will consider in detail the case of Markov semigroups associated with classical orthogonal polynomials–namely, the Ornstein-Uhlenbeck semigroup, the Laguerre semigroup and the Jacobi semigroup. We use the approach to analyze each of these semigroups. This approach h help to compare the semigroups, and see clearly their similarities and differences. Since the Ornstein-Uhlenbeck semigroup has wider attention due to its applications in Quantum Physics, we focus on it, as well as on its subordinated semigroup, the Poisson-Hermite semigroup. Finally, in section 4, we prove that the Ornstein-Uhlenbeck semigroup is not only a contraction semigroup, but that it is also hypercontractive. In order to do that we will prove, as it was done originally by Leonard Gross [**40**], that the Ornstein-Uhlenbeck operator satisfies a logarithmic Sobolev inequality and then that the hypercontractivity property of the semigroup is equivalent to that inequality. Then we will study functional inequalities, which relate the L^p norm of a function to the L^q norm of its (weak) gradient. The functional inequalities that we are going to study in detail are: Sobolev inequalities, logarithmic Sobolev inequalities, and spectral gap inequalities. We will analyze their characteristics, their properties and the relations among them. We will also consider curvature-dimension inequalities which are a development of the famous Bakry-Emery criteria for hypercontractivity of a semigroup [**13**]. These inequalities have important applications in Differential Geometry and make it possible to analyze the local structure of differential operators. As we will see, the relationship between curvature-dimension inequalities and functional inequalities will be crucial in establishing the results in the last part of the notes.

Since these are lecture notes intended for graduate students, we have tried to make them as self-contained as possible. Thus, we have provided full details for most of the proofs. For the same reason, several appendixes are included. The first one considers the Gamma function and related functions. The second one contains the main properties and formulas of the classical orthogonal polynomials. In the final one, we study the classical semigroups in Analysis (the heat semigroup and the Poisson semigroup). This study will make easier the comparison between them and the semigroups considered in section 2.

The origin of these notes goes back to a seminar we gave during the first semester 2004, at Escuela de Matemáticas de la Facultad de Ciencias Universidad Central de Venezuela (UCV) on the D. Bakry's monograph [**12**]. I want to thank all the participants in the seminar for their enthusiasm and perseverance. In addition, the talks we gave at the analysis seminar of the Universidad Central de Venezuela, the Universidad de los Andes, Université d' Angers, the University of Kansas and the University of Missouri, Columbia, helped us to shape these notes. I want to thank Piotr Graczyk for several conversations on the hypercontractivity property. I am grateful to my students Cristina Balderrama and Ebner Pineda for their careful reading and countless corrections of previous versions of these notes. I must say that their suggestions have contributed to the improvement of the quality and clarity of the notes, in a remarkable way. All the remaining errors are my whole responsibility.

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2. Markov Semigroups

2.1. Basic Definitions. — A one-parameter semigroup of operators $\{T_t\}_{t\geq 0}$ on a Banach space X is a family of operators indexed by non-negative real numbers such that

i) $T_0 = I$, the identity operator in X.

ii) $T_{s+t} = T_s \circ T_t, \quad$ for all $t, s \geq 0$.

We are going to develop the theory of operator semigroups from the point of view of probability theory and for that reason we are going to focus our study on Markov semigroups in a probability space $(E, \mathcal{B}, \mu)$.

Definition 2.1. — A family of *transition probabilities* $\{P_t(x, dy)\}$ is a family of kernels $P_t(x, dy)$ such that for all $t \geq 0$

i) $P_t(\cdot, B)$ is a measurable function, for each $B \in \mathcal{B}$.
ii) $P_t(x, \cdot)$ is a probability measure on $(E, \mathcal{B})$, for $x \in E$.

Definition 2.2. — A *Markov semigroup* on E is a family of transition probability kernels $\{P_t(x, dy)\}$ that satisfy the following properties:

i) $P_0(x, dy)$ is the unit mass at x
ii) The Chapman-Kolmogorov identity holds

$$\int_E P_s(x, dy) P_t(y, dz) = P_{s+t}(x, dz). \tag{2.1}$$

Markov semigroups appear naturally in the study of Markov processes, where the probability measure $P_t(x, dy)$ is the law of a Markov process $\{X_t\}$, with values in E, starting from the point x at time 0. More precisely, given Markov semigroup on E a Markov process $\{X_t\}$ can be constructed by defining its finite-dimensional distributions

$$\begin{aligned} &Prob\{X_{t_1} \in B_1, X_{t_2} \in B_1, \quad \cdots, X_{t_k} \in B_k | X_0 = x\} = \\ &\int_{B_k} \cdots \int_{B_2} \int_{B_1} \quad P_{t_1}(x, dy_1) P_{t_2 - t_1}(y_1, dy_2) \cdots P_{t_k - t_{k-1}}(y_{k-1}, dy_k). \end{aligned} \tag{2.2}$$

However, we are not going to consider Markov processes, and we refer to **[19]** or **[50]**, and also to the notes of P. Graczyk in this volume for details.

Our main interest is the family of *Markov operators* $\{T_t\}_{t \geq 0}$ defined on the space of positive or bounded Borel-measurable functions on E by

$$T_t f(x) = \int_E f(y)\, P_t(x, dy). \tag{2.3}$$

This family can be identified then with the Markov semigroup $\{P_t(x, dy)\}$ in a natural way and entirely characterizes it. We also require that, for each function $f \in L^2(\mu) = L^2(E, \mathcal{B}, \mu)$,

$$\lim_{t \to 0^+} T_t f = f,$$

where the limit is in the $L^2(\mu)$ sense.

By the Chapman-Kolmogorov identity for $\{P_t(x,dy)\}$ (2.1), it is clear that the family $\{T_t\}$ is a semigroup of operators, since

$$\begin{aligned}(T_t \circ T_s)f(x) &= T_t(\int_E f(y)P_s(x,dy)) = \int_E\int_E f(y)P_s(u,dy)P_t(x,du)\\ &= \int_E f(y)\int_E P_s(u,dy)P_t(x,du) = \int_E f(y)P_{t+s}(x,dy) = T_{t+s}f(x).\end{aligned}$$

Moreover, by property ii) for transition probabilities, the operator T_t clearly preserves positivity, that is to say if $f \geq 0$ then

$$T_t f(x) = \int_E f(y)P_t(x,dy) \geq 0,$$

since $P_t(x,dy)$ is a (positive) probability measure and for the same reason T_t is *conservative*,

$$T_t 1 = 1. \tag{2.4}$$

In general, we will say that a semigroup of operators $\{T_t\}_{t\geq 0}$ satisfies the *Markov property,* if it is conservative and preserves positivity. In that case we will call $\{T_t\}_{t\geq 0}$ a Markov (operator) semigroup.

Since T_t is given in terms of a probability measure, by Jensen's inequality, we have for any convex function ϕ,

$$T_t(\phi \circ f) \geq \phi(T_t f). \tag{2.5}$$

If we consider the Markov process $\{X_t\}$ on E associated with the Markov semigroup $\{P_t(x,dy)\}$ on E, then we can represent the latter as

$$T_t f(x) = \mathbf{E}[f(X_t)|X_0 = x]. \tag{2.6}$$

Definition 2.3. — We say a probability measure μ of the probability space $(E,\mathcal{B},\mu)$ is an *invariant measure* (or stationary measure) for the semigroup $\{T_t\}$, if

$$\int_E T_t f\,d\mu = \int_E f\,d\mu, \tag{2.7}$$

for any positive function $f \in L^1(\mu) = L^1(E,\mathcal{B},\mu)$.

In most cases, the invariant measure of a semigroup $\{T_t\}_{t\geq 0}$ is unique, up to a multiplicative constant. If the measure μ is finite we always normalize it so that it is a probability measure.

Let us observe that if μ is an invariant measure for the semigroup $\{T_t\}$, then for any $f \in L^1(\mu)$, we have

$$\begin{aligned}\int_E T_t f\,d\mu &= \int_E T_t(f^+ - f^-)\,d\mu = \int_E (T_t f^+ - T_t f^-)\,d\mu\\ &= \int_E (f^+ - f^-)\,d\mu = \int_E f\,d\mu.\end{aligned}$$