Séminaires & Congrès
25, 2012, p. 61–110

RANDOM MATRICES AND ORTHOGONAL POLYNOMIALS

by

Jacques Faraut

Abstract. — The central question of the theory of random matrices is to determine the asymptotic behavior of the eigenvalues of large random symmetric or Hermitian matrices. In the case of the Gaussian Unitary Ensemble (GUE), i.e. the space of Hermitian matrices equipped with a unitarily invariant Gaussian probability, Mehta's formulae express the eigenvalue density in terms of the Christoffel-Darboux kernel of the Hermite polynomials. In fact orthogonal polynomials are a powerful tool in this theory. We will present in this course methods in the theory of random matrices which are using orthogonal polynomials.

***Résumé* (Matrices aléatoires et polynômes orthogonaux).** — La question centrale de la théorie des matrices aléatoires est de déterminer le comportement asymptotique des valeurs propres d'une matrice symétrique ou hermitienne de grande dimension. Dans le cas de l'Ensemble Unitaire Gaussien (GUE), c'est-à-dire l'espace des matrices hermitiennes muni d'une probabilité gaussienne invariante par le groupe unitaire, les formules de Mehta expriment la densité des valeurs propres à l'aide du noyau de Christoffel-Darboux des polynômes d'Hermite. En effet les polynômes orthogonaux sont un outil puissant dans cette théorie. Nous présenterons dans ce cours des méthodes de la théorie des matrices aléatoires qui utilisent les polynômes orthogonaux.

1. Introduction

For $\mathbb{F} = \mathbb{R}, C$ or $\mathbb{H}$, let $H_n = Herm(n, \mathbb{F})$ be the space of $n \times n$ Hermitian matrices with entries in $\mathbb{F}$. On H_n one considers the probability law defined by

$$\mathbb{P}_n(dx) = \frac{1}{C_n} \exp\bigl(-\gamma \mathrm{tr}\,(x^2)\bigr) m(dx),$$

2010 Mathematics Subject Classification. — 33C45; 42A38; 60B10.
Key words and phrases. — Random matrices, Wigner semi-circle law, Laguerre polynomials.

where γ is a positive parameter, m is the Euclidean measure associated with the inner product

$$(x|y) = \mathrm{tr}\,(xy),$$

and

$$C_n = \int_{H_n} \exp\bigl(-\gamma \mathrm{tr}\,(x^2)\bigr) m(dx) = \Bigl(\sqrt{\frac{\pi}{\gamma}}\Bigr)^N,$$

where

$$N = \dim_{\mathbb{R}} H_n = n + \frac{\beta}{2} n(n-1), \quad \beta = \dim_{\mathbb{R}} \mathbb{F} = 1, 2, 4.$$

This probability is invariant under the group $U_n = U(n; \mathbb{F})$ of $n \times n$ unitary matrices with entries in $\mathbb{F}$, acting on H_n by the transformations

$$x \mapsto uxu^* \quad (u \in U_n).$$

For $\mathbb{F} = \mathbb{R}$, it is the orthogonal group $O(n)$, for $\mathbb{F} = \mathbb{C}$ it is the unitary group $U(n)$, and for $\mathbb{F} = \mathbb{H}$, it is isomorphic to the symplectic group $Sp(n)$, maximal compact subgroup of the complex symplectic group $Sp(n, \mathbb{C})$.

The probability space $(H_n, \mathbb{P}_n)$ is called *Gaussian Orthogonal Ensemble* (GOE) for $\mathbb{F} = \mathbb{R}$, *Gaussian Unitary Ensemble* (GUE) for $\mathbb{F} = \mathbb{C}$, and *Gaussian Symplectic Ensemble* (GSE) for $\mathbb{F} = \mathbb{H}$.

The general problem in the theory of random matrices is to study asymtotics of probabilities related to the eigenvalues of a random matrix for large n.

1.1. Statistical distribution of the eigenvalues. — If $B \subset \mathbb{R}$ is a Borel set, one denotes by $\xi_{n,B}$ the random variable defined by

$$\xi_{n,B}(x) = \frac{1}{n} \#\{\text{eigenvalues of } x \text{ in } B\}.$$

Let $\mu_n(B)$ be its expectation,

$$\mu_n(B) = \mathbb{E}_n(\xi_{n,B}).$$

Then μ_n is a probability measure on $\mathbb{R}$, it is the statistical distribution of the eigenvalues. If χ_B is the characteristic function of the set B, then

$$\xi_{n,B}(x) = \frac{1}{n}\bigl(\chi_B(\lambda_1) + \cdots + \chi_B(\lambda_n)\bigr),$$

where $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of x. In the sense of functional calculus this can be written

$$\xi_{n,B}(x) = \frac{1}{n} \mathrm{tr}\, \chi_B(x).$$

Therefore

$$\mu_n(B) = \frac{1}{n} \int_{H_n} \mathrm{tr}\, \chi_B(x) \mathbb{P}_n(x).$$

More generally, if φ is a bounded measurable function on $\mathbb{R}$,

$$\int_{\mathbb{R}} \varphi(t) \mu_n(dt) = \frac{1}{n} \int_{H_n} \mathrm{tr}\,\bigl(\varphi(x)\bigr) \mathbb{P}_n(dx).$$

Question : what can be said about the asymptotics of μ_n as n goes to infinity ? The answer is given by the following theorem of Wigner.

The semi-circle law σ_a of radius a is the probability measure defined on $\mathbb{R}$ by

$$\int_{\mathbb{R}} \varphi(t)\sigma_a(dt) = \frac{2}{\pi a^2}\int_{-a}^{a} \varphi(t)\sqrt{a^2-t^2}dt.$$

The theorem of Wigner says that, after scaling, the measure μ_n converges to the semi-circle law σ_a of radius

$$a = \sqrt{\frac{\beta}{\gamma}}.$$

***Theorem 1.1* (Wigner).** — *Let φ be a bounded continuous function on $\mathbb{R}$. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}} \varphi\Big(\frac{t}{\sqrt{n}}\Big)\mu_n(dt) = \frac{2}{\pi a^2}\int_{-a}^{a} \varphi(u)\sqrt{a^2-u^2}du.$$

This means that, for large n, the density of eigenvalues is approximatively

$$\frac{2}{\pi a^2}\sqrt{na^2-\lambda^2},$$

if $|\lambda| \leq a\sqrt{n}$, and 0 if $|\lambda| \geq a\sqrt{n}$.

In the original proof Wigner considers the moments of the measure μ_n:

$$\mathfrak{M}_k(\mu_n) = \int_{\mathbb{R}} t^k \mu_n(dt) = \frac{1}{n}\int_{H_n} \operatorname{tr}(x^k)\mathbb{P}_n(dx),$$

and by combinatorial computations determines the asymptotics of $\mathfrak{M}_k(\mu_n)$ as n goes to infinity: for k fixed,

$$\mathfrak{M}_{2k}(\mu_n) \sim \Big(\frac{\beta}{4\gamma}\Big)^k \frac{(2k)!}{k!(k+1)!}n^k.$$

Note that the moments of odd order vanish. On the other hand it is easy to compute the moments of the semi-circle law:

$$\mathfrak{M}_{2k}(\sigma_a) = \Big(\frac{a^2}{4}\Big)^k \frac{(2k)!}{k!(k+1)!}.$$

In fact

$$\begin{aligned}\mathfrak{M}_{2k}(\sigma_a) &= \frac{2}{\pi a^2}\int_{-a}^{a} t^{2k}\sqrt{a^2-t^2}dt = \frac{2a^{2k}}{\pi}\int_0^1 u^{k-\frac{1}{2}}\sqrt{1-u}du \\ &= \frac{2a^{2k}}{\pi}B\Big(k+\frac{1}{2},\frac{3}{2}\Big) = \frac{2a^{2k}}{\pi}\frac{\Gamma\big(k+\frac{1}{2}\big)\Gamma\big(\frac{3}{2}\big)}{\Gamma(k+2)} = \frac{a^{2k}}{2^{2k}}\frac{(2k)!}{k!(k+1)!}.\end{aligned}$$

The proof by Pastur uses the Cauchy transform. Recall that the Cauchy transform of a probability measure μ on $\mathbb{R}$ is the function G_μ defined on $\mathbb{C}\setminus\mathbb{R}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\mu(dt).$$

For $\mu = \mu_n$, writing $G_{\mu_n} = G_n$,

$$G_n(z) = \frac{1}{n} \int_{H_n} \operatorname{tr}\big((zI - x)^{-1}\big)\mathbb{P}_n(dx).$$

After scaling one has to look at the functions

$$\tilde{G}_n(z) = \sqrt{n} G_n(\sqrt{n} z).$$

The proof amounts to showing that the functions $\tilde{G}_n$ converge,

$$\lim_{n\to\infty} \tilde{G}_n(z) = f(z),$$

and that the limit f is a holomorphic function satisfying

$$f(z)^2 - \frac{4}{a^2} z f(z) + \frac{4}{a^2} = 0.$$

Since $\Im G_n(z) < 0$ and hence $\Im f(z) < 0$ for $\Im z > 0$, necessarily

$$f(z) = \frac{2}{a^2}\big(z - \sqrt{z^2 - a^2}\big),$$

which is the Cauchy transform of the semi-circle law σ_a.

The proof we will present uses the Fourier transform,

$$\widehat{\mu_n}(\tau) = \int_{\mathbb{R}} e^{-it\tau} \mu_n(t) = \frac{1}{n} \int_{H_n} \operatorname{tr}\big(\exp(-i\tau x)\big)\mathbb{P}_n(dx).$$

We will see that it can be computed in terms of Laguerre polynomials. The convergence to the semi-circle law will follow by using the classical Lévy-Cramér theorem.

More general results are obtained by using logarithmic potential theory. One defines the energy of a probability measure μ on $\mathbb{R}$ by

$$I(\mu) = \int_{\mathbb{R}^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}} V(t)\mu(dt).$$

For $V(t) = \gamma t^2$, the semi-circle law appears as equilibrium measure: measure which realizes the minimum of the energy.

1.2. Local behaviour : the probabilities $A_n(m,\theta)$. — For $\theta > 0$, and $0 \leq m \leq n$, one denotes by $A_n(m,\theta)$ the probability that a matrix $x \in H_n$ has m eigenvalues in the interval $[-\theta,\theta]$. By using orthogonal polynomials one can evaluate the probability $A_n(m,\theta)$ in terms of Fredholm determinants, and its behaviour as $n \to \infty$. In particular we will see that, for $m = 0$,

$$\lim_{n\to\infty} A_n\Big(0, \frac{\theta}{\sqrt{2n}}\Big) = \operatorname{Det}_{[-\theta,\theta]}(I - \mathcal{K}),$$

where Det is the Fredholm determinant, and $\mathcal{K}$ is the kernel

$$\mathcal{K}(\xi,\eta) = \frac{1}{\pi} \frac{\sin(\xi - \eta)}{\xi - \eta},$$

restricted to the square $[-\theta,\theta] \times [-\theta,\theta]$.

1.3. Wishart Unitary Ensemble. — In the last chapter we consider the *Wishart Unitary Ensemble*. In that case there is an analogue of Wigner Theorem: It is Marchenko-Pastur Theorem which describes the asymptotic of the statistical distribution of the eigenvalues for a Wishart random matrix.

2. Orthogonal polynomials

2.1. Heine's formulae. — Let μ be a positive measure on $\mathbb{R}$. We assume that the support of μ is infinite, and that, for all $m \geq 0$,

$$\int_{\mathbb{R}} |t|^m \mu(dt) < \infty.$$

Hence, for all $j \in \mathbb{N}$, the moment of order j,

$$m_j = \int_{\mathbb{R}} t^j \mu(dt),$$

is defined. On the space $\mathcal{P}$ of polynomials in one variable with real coefficients one considers the inner product

$$(p|q) = \int_{\mathbb{R}} p(t)q(t)\mu(dt),$$

for which $\mathcal{P}$ is a pre-Hilbert space. The monomials $1, t, \ldots, t^m, \ldots$ are independent, and, by the Gram-Schmidt orthogonalization, one gets a sequence $\{p_m\}$ of orthogonal polynomials: p_m is of degree m, and

$$\int_{\mathbb{R}} p_m(t)p_n(t)\mu(dt) = 0 \text{ if } m \neq n.$$

If $\{p_m\}$ is a sequence of orthogonal polynomials we will write

$$p_m(t) = a_m t^m + \cdots,$$

$$d_m = \int_{\mathbb{R}} p_m(t)^2 \mu(dt).$$

Example: Hermite polynomials. The measure μ is Gaussian :

$$\mu(dt) = e^{-t^2} dt.$$

The Hermite polynomial H_m is defined by

$$H_m(t) = (-1)^m e^{t^2} \left(\frac{d}{dt}\right)^m e^{-t^2}.$$

Notice that $a_m = 2^m$. By integrating by parts one shows that

$$d_m = 2^m m! \sqrt{\pi}.$$

In fact, for any polynomial p,

$$\int_{\mathbb{R}} H_m(t)p(t)e^{-t^2} dt = \int_{\mathbb{R}} p^{(m)}(t)e^{-t^2} dt,$$