Séminaires & Congrès

COLLECTION SMF

OPERADS OF NATURAL OPERATIONS I

Michael Batanin & Clemens Berger & Martin Markl

OPERADS 2009

Numéro 26

Jean-Louis Loday & Bruno Vallette, ed.

SOCIÉTE MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
26, 2011, p. 1–33

OPERADS OF NATURAL OPERATIONS I: LATTICE PATHS, BRACES AND HOCHSCHILD COCHAINS

by

Michael Batanin, Clemens Berger & Martin Markl

Abstract. — In this first paper of a series we study various operads of natural operations on Hochschild cochains and relationships between them.

Résumé **(Opérades des opérations naturelles I: chemins brisés, opérations brace et cochaînes de Hochschild)**

Dans ce premier article d'une série nous étudions et comparons plusieurs opérades munies d'une action naturelle sur les cochaines de Hochschild d'une algèbre associative.

1. Introduction

This paper continues the efforts of [**14**, **3**, **2**] in which we studied operads naturally acting on Hochschild cochains of an associative or symmetric Frobenius algebra. A general approach to the operads of natural operations in algebraic categories was set up in [**14**] and the first breakthrough in computing the homotopy type of such an operad has been achieved in [**3**]. In [**2**], the same problem was approached from a combinatorial point of view, and a machinery which produces operads acting on the Hochschild cochain complex in a general categorical setting was introduced.

The constructions of [**2**] have some specific features in different categories which are important in applications. In this first paper of a series entitled 'Operads of Natural Operations' we begin a detailed study of these special cases.

2000 Mathematics Subject Classification. — Primary 18D50, secondary 55P48, 55R80.
Key words and phrases. — Lattice path operad, Hochschild cohomology, natural operation.

M. Batanin was supported by Scott Russell Johnson Memorial Fund and Australian Research Council Grant DP0558372.
C. Berger was supported by the grant OBTH of the French Agence Nationale de Recherche.
M. Markl was supported by the grant GA ČR 201/08/0397 and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

It is very natural to start with the classical Hochschild cochain complex of an associative algebra. This is, by far, the most studied case. It seems to us, however, that a systematic treatment is missing despite its long history and a vast amount of literature available. One of the motivations of this paper was our wish to relate various approaches in literature and to provide a uniform combinatorial language for this purpose.

Here is a short **summary** of the paper.

In section 2 we describe our main combinatorial tool: the lattice path operad $\mathcal{L}$ and its condensation in the differential graded setting. This description leads to a careful treatment of (higher) brace operations and their relationship with lattice paths in section 3.

The lattice path operad comes equipped with a filtration by complexity [2]. The second filtration stage $\mathcal{L}_{(2)}$ is the most important for understanding natural operations on the Hochschild cochains. In section 4 we give an alternative description of $\mathcal{L}_{(2)}$ in terms of trees, closely related to the operad of natural operations from [14]. Finally, in section 5 we study various suboperads generated by brace operations. The main result is that **all these operads have the homotopy type of a chain model of the little disks operad**. For sake of completeness we add a brief appendix containing an overview of some categorical constructions used in this paper.

Convention. If not stated otherwise, by an *operad* we mean a classical symmetric (i.e. with the symmetric groups acting on its components) operad in an appropriate symmetric monoidal category which will be obvious from the context. The same convention is applied to coloured operads, substitudes, multitensors and functor-operads recalled in the appendix.

Acknowledgement. We would like to express our thanks to the referee for carefully reading the paper and many useful remarks and suggestions.

2. The lattice path operad

As usual, for a non-negative integer m, $[m]$ denotes the ordinal $0 < \cdots < m$. We will use the same symbol also for the category with objects $0, \ldots, m$ and the unique morphism $i \to j$ if and only if $i \leq j$. The *tensor product* $[m] \otimes [n]$ is the category freely generated by the (m,n)-grid which is, by definition, the oriented graph with vertices (i,j), $0 \leq i \leq m$, $0 \leq j \leq n$, and one oriented edge $(i',j') \to (i'',j'')$ if and only if $(i'',j'') = (i'+1,j')$ or $(i'',j'') = (i',j'+1)$.

Let us recall, closely following [2], the *lattice path operad* and its basic properties. For non-negative integers $k_1, \ldots, k_n, l$ and $n \in \mathbb{N}$ put

$$\mathcal{L}(k_1, \ldots, k_n; l) := Cat_{*,*}([l+1], [k_1+1] \otimes \cdots \otimes [k_n+1])$$

where $\otimes$ is the tensor product recalled above and $Cat_{*,*}([l+1], [k_1+1] \otimes \cdots \otimes [k_n+1])$ the set of functors φ that preserve the extremal points, by which we mean that

$$\varphi(0) = (0, \ldots, 0) \ \text{ and } \ \varphi(l+1) = (k_1+1, \ldots, k_n+1). \tag{1}$$

A functor $\varphi \in \mathcal{L}(k_1, \dots, k_n; l)$ is given by a chain of $l+1$ morphisms $\varphi(0) \to \varphi(1) \to \cdots \to \varphi(l+1)$ in $[k_1+1] \otimes \cdots \otimes [k_n+1]$ with $\varphi(0)$ and $\varphi(l+1)$ fulfilling (1). Each morphism $\varphi(i) \to \varphi(i+1)$ is determined by a finite oriented edge-path in the $(k_1+1, \dots, k_n+1)$-grid. For $n = 0$, $\mathcal{L}(;l)$ consists of the unique functor from $[l+1]$ to the terminal category with one object.

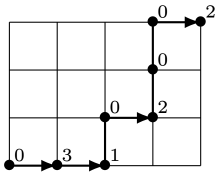
2.1. Marked lattice paths. — We will use a slight modification of the terminology of [2]. For non-negative integers $k_1, \dots, k_n \in \mathbb{N}$ denote by $\mathcal{Q}(k_1, \dots, k_n)$ the integral hypercube

$$\mathcal{Q}(k_1, \dots, k_n) := [k_1+1] \times \cdots \times [k_n+1] \subset \mathbb{Z}^{\times n}.$$

A *lattice path* is a sequence $p = (x_1, \dots, x_N)$ of $N := k_1 + \cdots + k_n + n + 1$ points of $\mathcal{Q}(k_1, \dots, k_n)$ such that x_{a+1} is, for each $0 \le a < N$, given by increasing exactly one coordinate of x_a by 1. A *marking* of p is a function $\mu : p \to \mathbb{N}$ that assigns to each point x_a of p a non-negative number $\mu_a := \mu(x_a)$ such that $\sum_{a=1}^{N} \mu_a = l$.

We can describe functors in $\mathcal{L}(k_1, \dots, k_n; l)$ as marked lattice paths (p, μ) in the hypercube $\mathcal{Q}(k_1, \dots, k_n)$. The marking $\mu_a = \mu(x_a)$ represents the number of elements of the interior $\{1, \dots, l\}$ of $[l+1]$ that are mapped by φ to the ath lattice point x_a of p. We call lattice points marked by 0 *unmarked* points so the set of marked points equals $\varphi(\{1, \dots, l\})$. For example, the marked lattice path

(2)



represents a functor $\varphi \in \mathcal{L}(3,2;8)$ with $\varphi(0) = (0,0)$, $\varphi(1) = \varphi(2) = \varphi(3) = (1,0)$, $\varphi(4) = (2,0)$, $\varphi(5) = \varphi(6) = (3,1)$ and $\varphi(7) = \varphi(8) = \varphi(9) = (4,3)$. The lattice is trivial for $n = 0$, so the unique element of $\mathcal{L}(;l)$ is represented by the point marked l, i.e. by $\bullet^l$.

2.2. Definition. — Let $p \in \mathcal{L}(k_1, \dots, k_n; l)$ be a lattice path. A point of p at which p changes its direction is an *angle* of p. An *internal point* of p is a point that is not an angle nor an extremal point of p. We denote by $Angl(p)$ (resp. $Int(p)$) the set of all angles (resp. internal points) of p.

For instance, the path in (2) has 4 angles, 2 internal points, 4 unmarked points and 1 unmarked internal point.

Following again [2] closely, we denote, for $1 \le i < j \le n$, by p_{ij} the projection of the path $p \in \mathcal{L}(k_1, \dots, k_n; l)$ to the face $[k_i+1] \times [k_j+1]$ of $\mathcal{Q}(k_1, \dots, k_n)$; let $c_{ij} := \#Angl(p_{ij})$ be the number of its angles. The maximum $c(p) := \max\{c_{ij}\}$ is called the *complexity* of p. Let us finally denote by $\mathcal{L}_{(c)}(k_1, \dots, k_n; l) \subset \mathcal{L}(k_1, \dots, k_n; l)$ the subset of marked lattice paths of complexity $\le c$. The case $c = 2$ is particularly interesting, because $\mathcal{L}_{(2)}(k_1, \dots, k_n; l)$ is, by [2, Proposition 2.14], isomorphic to the space of unlabeled $(l; k_1, \ldots, k_n)$-trees recalled on page 19. For convenience of the reader we recall this isomorphism on page 20.

As shown in [2], the sets $\mathfrak{L}(k_1, \ldots, k_n; l)$ and their subsets $\mathfrak{L}_{(c)}(k_1, \ldots, k_n; l)$, $c \geq 0$, form an $\mathbb{N}$-coloured operad $\mathfrak{L}$ and its sub-operads $\mathfrak{L}_{(c)}$. To simplify formulations, we will allow $c = \infty$, putting $\mathfrak{L}_{(\infty)} := \mathfrak{L}$.

2.3. Convention. — Since we aim to work in the category of abelian groups, we will make no notational difference between the sets $\mathfrak{L}_{(c)}(k_1, \ldots, k_n; l)$ and their linear spans.

The underlying category of the coloured operad $\mathfrak{L}$ (which coincides with the underlying category of $\mathfrak{L}_{(c)}$ for any $c \geq 0$) is, by definition, the category whose objects are non-negative integers and morphism $n \to m$ are elements of $\mathfrak{L}(n, m)$, i.e. non-decreasing maps $\varphi : [m+1] \to [n+1]$ preserving the endpoints.

By Joyal's duality [12], this category is isomorphic to the (skeletal) category Δ of finite ordered sets, i.e. $\mathfrak{L}(n, m) = \Delta(n, m)$. The operadic composition makes the collection $\mathfrak{L}_{(c)}(\bullet_1, \ldots, \bullet_n; \bullet)$ (with $c = \infty$ allowed) a functor $(\Delta^{\mathrm{op}})^{\times n} \times \Delta \to \mathrm{Abel}$, i.e. n-times simplicial 1-time cosimplicial Abelian group.

Morphisms in the category Δ are generated by the cofaces $d_i : [m-1] \to [m]$ given by the non-decreasing map that misses i, and the codegeneracies $s_i : [m+1] \to [m]$ given by the non-decreasing map that hits i twice. In both cases, $0 \leq i \leq m$. Let us inspect how these generating maps act on the pieces of the operad $\mathfrak{L}_{(c)}$.

2.4. Simplicial structures. — We describe the induced rth $(1 \leq r \leq n)$ simplicial maps

$$\partial_i^r : \mathfrak{L}_{(c)}(k_1, \ldots, k_{r-1}, m, k_{r+1}, \ldots, k_n; l) \to \mathfrak{L}_{(c)}(k_1, \ldots, k_{r-1}, m-1, k_{r+1}, \ldots, k_n; l),$$

where $m \geq 1$, $0 \leq i \leq m$, and

$$\sigma_i^r : \mathfrak{L}_{(c)}(k_1, \ldots, k_{r-1}, m, k_{r+1}, \ldots, k_n; l) \to \mathfrak{L}_{(c)}(k_1, \ldots, k_{r-1}, m+1, k_{r+1}, \ldots, k_n; l),$$

where $0 \leq i \leq m$. To this end, we define, for each $m \geq 1$ and $0 \leq i \leq m$, the epimorphism of the hypercubes

$$D_i^r : \mathfrak{Q}(k_1, \ldots, k_{r-1}, m, k_{r+1}, \ldots, k_n) \twoheadrightarrow \mathfrak{Q}(k_1, \ldots, k_{r-1}, m-1, k_{r+1}, \ldots, k_n)$$

by

$$D_i^r(a_1, \ldots, a_r, \ldots, a_n) := \begin{cases} (a_1, \ldots, a_r, \ldots, a_n), & \text{if } a_r \leq i, \text{ and} \\ (a_1, \ldots, a_r - 1, \ldots, a_n), & \text{if } a_r > i, \end{cases}$$

where $(a_1, \ldots, a_r, \ldots, a_n) \in \mathfrak{Q}(k_1, \ldots, k_{r-1}, m, k_{r+1}, \ldots, k_n)$ is an arbitrary point. In a similar fashion, the monomorphism

$$S_i^r : \mathfrak{Q}(k_1, \ldots, k_{r-1}, m, k_{r+1}, \ldots, k_n) \hookrightarrow \mathfrak{Q}(k_1, \ldots, k_{r-1}, m+1, k_{r+1}, \ldots, k_n)$$

is, for $0 \leq i \leq m$, given by

$$S_i^r(a_1, \ldots, a_r, \ldots, a_n) := \begin{cases} (a_1, \ldots, a_r, \ldots, a_n), & \text{if } a_r \leq i, \text{ and} \\ (a_1, \ldots, a_r + 1, \ldots, a_n), & \text{if } a_r > i. \end{cases}$$