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Yves Guiraud & Philippe Malbos

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# IDENTITIES AMONG RELATIONS FOR HIGHER-DIMENSIONAL REWRITING SYSTEMS

*by*

Yves Guiraud & Philippe Malbos

***Abstract.*** — We generalize the notion of identities among relations, well known for presentations of groups, to presentations of n-categories by polygraphs. To each polygraph, we associate a track n-category, generalizing the notion of crossed module for groups, in order to define the natural system of identities among relations. We relate the facts that this natural system is finitely generated and that the polygraph has finite derivation type.

***Résumé*** **(Identités entre les relations pour la réécriture en dimension supérieure)**

Nous généralisons la notion d'identités entre les relations, bien connue pour les présentations de groupes, aux présentations de n-catégories par polygraphes. À chaque polygraphe, nous associons une track n-catégorie, généralisant la notion de module croisé pour les groupes, afin de définir son système naturel des identités entre les relations. Nous relions le fait que ce système naturel soit de type fini avec le fait que le polygraphe soit de type de dérivation fini.

## Introduction

The notion of *identity among relations* originates in the work of Peiffer and Reidemeister, in combinatorial group theory [14, 17]. It is based on the notion of *crossed module*, introduced by Whitehead, in algebraic topology, for the classification of homotopy 2-types [20, 21]. Crossed modules have also been defined for other algebraic structures than groups, such as commutative algebras [16], Lie algebras [11] or categories [15]. Then Baues has introduced *track 2-categories*, which are categories enriched in groupoids, as a model of homotopy 2-type [1, 2], together with *linear track extensions*, as generalizations of crossed modules [4].

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There exist several interpretations of identities among relations for presentations of groups: as homological 2-syzygies [5], as homotopical 2-syzygies [12] or as Igusa's pictures [10, 12]. One can also interpret identities among relations as the critical pairs of a group presentation by a convergent word rewriting system [7]. This point of view yields an algorithm based on Knuth-Bendix's completion procedure that computes a family of generators of the module of identities among relations [9].

In this work, we define the notion of identities among relations for $n$-categories presented by higher-dimensional rewriting systems called *polygraphs* [6], using notions introduced in [8]. Given an $n$-polygraph $\Sigma$, we consider the free *track $n$-category* $\Sigma^{\top}$ generated by $\Sigma$, that is, the free $(n-1)$-category enriched in groupoid on $\Sigma$. We define *identities among relations for* $\Sigma$ as the elements of an *abelian natural system* $\Pi(\Sigma)$ on the $n$-category $\overline{\Sigma}$ it presents. For that, we extend a result proved by Baues and Jibladze [3] for the case $n = 2$.

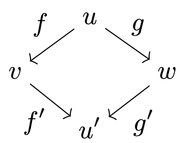
**Theorem 2.2.2.** *A track $n$-category $\mathcal{T}$ is abelian if and only if there exists a unique (up to isomorphism) abelian natural system $\Pi(\mathcal{T})$ on $\overline{\mathcal{T}}$ such that $\widehat{\Pi(\mathcal{T})}$ is isomorphic to $\mathrm{Aut}^{\mathcal{T}}$.*

We define $\Pi(\Sigma)$ as the abelian natural system associated by that result to the abelianized track $n$-category $\Sigma^{\top}_{\mathrm{ab}}$. In Section 2.2, we give an explicit description of $\Pi(\Sigma)$.

Then, in Section 2.4, we interpret generators of $\Pi(\Sigma)$ as elements of a *homotopy basis* of the track $n$-category $\Sigma^{\top}$, see [8]. More precisely, we prove:

**Theorem 2.4.1.** *If an $n$-polygraph $\Sigma$ has finite derivation type then the abelian natural system $\Pi(\Sigma)$ is finitely generated.*

To prove this result, we give a way to compute generators of $\Pi(\Sigma)$ from the critical pairs of a convergent polygraph $\Sigma$. Indeed, there exists, for every critical branching $(f, g)$ of $\Sigma$, a confluence diagram:




An $(n+1)$-cell filling such a diagram is called a *generating confluence* of $\Sigma$. It is proved in [8] that the generating confluences of $\Sigma$ form a homotopy basis of $\Sigma^{\top}$. We show here that they also form a generating set for the natural system $\Pi(\Sigma)$ of identities among relations.

## 1. Preliminaries

In this section, we recall several notions from [8]: presentations of $n$-categories by polygraphs (1.1), rewriting properties of polygraphs (1.2), track $n$-categories and homotopy bases (1.3).

**1.1. Higher-dimensional categories and polygraphs.** — We fix an $n$-category $\mathcal{C}$ throughout this section.

**1.1.1. *Notations.*** — We denote by $\mathcal{C}_k$ the set (and the $k$-category) of $k$-cells of $\mathcal{C}$. If $f$ is in $\mathcal{C}_k$, then $s_i(f)$ and $t_i(f)$ respectively denote the $i$-source and $i$-target of $f$; we drop the suffix $i$ when $i = k-1$. The source and target maps satisfy the *globular relations*:

$$(1) \qquad s_i s_{i+1} = s_i t_{i+1} \qquad \text{and} \qquad t_i s_{i+1} = t_i t_{i+1}.$$

If $f$ and $g$ are $i$-composable $k$-cells, that is when $t_i(f) = s_i(g)$, we denote by $f \star_i g$ their $i$-composite $k$-cell. We also write $fg$ instead of $f \star_0 g$. The compositions satisfy the *exchange relations* given, for every $i \neq j$ and every possible cells $f$, $g$, $h$ and $k$, by:

$$(2) \qquad (f \star_i g) \star_j (h \star_i k) = (f \star_j h) \star_i (g \star_j k).$$

If $f$ is a $k$-cell, we denote by $1_f$ its identity $(k+1)$-cell and, by abuse, all the higher-dimensional identity cells it generates. When $1_f$ is composed with cells of dimension $k+1$ or higher, we simply denote it by $f$. A $k$-cell $f$ with $s(f) = t(f) = u$ is called a *closed* $k$-cell with *base point* $u$.

**1.1.2. *Spheres.*** — Let $\mathcal{C}$ be an $n$-category and let $k \in \{0, \ldots, n\}$. A *$k$-sphere of* $\mathcal{C}$ is a pair $\gamma = (f, g)$ of parallel $k$-cells of $\mathcal{C}$, that is, with $s(f) = s(g)$ and $t(f) = t(g)$; we call $f$ the *source* of $\gamma$ and $g$ its *target*. We denote by $\mathbf{S}\mathcal{C}$ the set of $n$-spheres of $\mathcal{C}$. An $n$-category is *aspherical* when all of its $n$-spheres have shape $(f, f)$.

**1.1.3. *Cellular extensions.*** — A *cellular extension of* $\mathcal{C}$ is a pair $\Gamma = (\Gamma_{n+1}, \partial)$ made of a set $\Gamma_{n+1}$ and a map $\partial : \Gamma_{n+1} \to \mathbf{S}\mathcal{C}$. By considering all the formal compositions of elements of $\Gamma$, seen as $(n+1)$-cells with source and target in $\mathcal{C}$, one builds the *free* $(n+1)$-*category generated by* $\Gamma$, denoted by $\mathcal{C}[\Gamma]$.

The *quotient of* $\mathcal{C}$ *by* $\Gamma$, denoted by $\mathcal{C}/\Gamma$, is the $n$-category one gets from $\mathcal{C}$ by identification of $n$-cells $s(\gamma)$ and $t(\gamma)$, for every $n$-sphere $\gamma$ of $\Gamma$. We usually denote by $\overline{f}$ the equivalence class of an $n$-cell $f$ of $\mathcal{C}$ in $\mathcal{C}/\Gamma$. We write $f \equiv_\Gamma g$ when $\overline{f} = \overline{g}$ holds.

**1.1.4. *Polygraphs.*** — We define $n$-polygraphs and free $n$-categories by induction on $n$. A 1-*polygraph* is a graph, with the usual notion of free category.

An $(n+1)$-*polygraph* is a pair $\Sigma = (\Sigma_n, \Sigma_{n+1})$ made of an $n$-polygraph $\Sigma_n$ and a cellular extension $\Sigma_{n+1}$ of the free $n$-category generated by $\Sigma_n$. The *free* $(n+1)$-*category generated by* $\Sigma$ and the $n$-*category presented by* $\Sigma$ are respectively denoted by $\Sigma^*$ and $\overline{\Sigma}$ and defined by:

$$\Sigma^* = \Sigma_n^*[\Sigma_{n+1}] \qquad \text{and} \qquad \overline{\Sigma} = \Sigma_n^*/\Sigma_{n+1}.$$

An $n$-polygraph $\Sigma$ is *finite* when each set $\Sigma_k$ is finite, $0 \leq k \leq n$. Two $n$-polygraphs whose presented $(n-1)$-categories are isomorphic are *Tietze-equivalent.* A property on $n$-polygraphs that is preserved up to Tietze-equivalence is *Tietze-invariant.*

An $n$-category $\mathcal{C}$ is *presented* by an $(n+1)$-polygraph $\Sigma$ when it is isomorphic to $\overline{\Sigma}$. It is *finitely generated* when it is presented by an $(n+1)$-polygraph $\Sigma$ whose underlying $n$-polygraph $\Sigma_n$ is finite. It is *finitely presented* when it is presented by a finite $(n+1)$-polygraph.

**1.1.5. *Example.*** — Let us consider the monoid $\mathbf{As} = \{a_0, a_1\}$ with unit $a_0$ and product $a_1 a_1 = a_1$. We see $\mathbf{As}$ as a (1-)category with one 0-cell $a_0$ and one non-degenerate 1-cell $a_1 : a_0 \to a_0$. As such, it is presented by the 2-polygraph $\Sigma_2$ with one 0-cell $a_0$, one 1-cell $a_1 : a_0 \to a_0$ and one 2-cell $a_2 : a_1 a_1 \Rightarrow a_1$. Thus $\mathbf{As}$ is finitely generated and presented. In what follows, we use graphical notations for those cells, where the 1-cell $a_1$ is pictured as a vertical "string" | and the 2-cell $a_2$ as .

**1.1.6. *Contexts and whiskers.*** — A *context of* $\mathcal{C}$ is a pair $(x, C)$ made of an $(n-1)$-sphere $x$ of $\mathcal{C}$ and an $n$-cell $C$ in $\mathcal{C}[x]$ such that $C$ contains exactly one occurrence of $x$. We denote by $C[x]$, or simply by $C$, such a context. If $f$ is an $n$-cell which is parallel to $x$, then $C[f]$ is the $n$-cell of $\mathcal{C}$ one gets by replacing $x$ by $f$ in $C$.

Every context $C$ of $\mathcal{C}$ has a decomposition

$$C \;=\; f_n \star_{n-1} (f_{n-1} \star_{n-2} (\cdots \star_1 f_1 x g_1 \star_1 \cdots) \star_{n-2} g_{n-1}) \star_{n-1} g_n,$$

where, for every $k$ in $\{1, \ldots, n\}$, $f_k$ and $g_k$ are $k$-cells of $\mathcal{C}$. A *whisker of* $\mathcal{C}$ is a context that admits such a decomposition with $f_n$ and $g_n$ being identities. Every context $C$ of $\mathcal{C}_{n-1}$ yields a whisker of $\mathcal{C}$ such that $C[f \star_{n-1} g] = C[f] \star_{n-1} C[g]$ holds.

If $\Gamma$ is a cellular extension of $\mathcal{C}$, then every non-degenerate $(n+1)$-cell $f$ of $\mathcal{C}[\Gamma]$ has a decomposition

$$f \;=\; C_1[\varphi_1] \star_n \cdots \star_n C_k[\varphi_k],$$

with $k \geq 1$ and, for every $i$ in $\{1, \ldots, k\}$, $\varphi_i$ in $\Gamma$ and $C_i$ a context of $\mathcal{C}$.

The *category of contexts of* $\mathcal{C}$ is denoted by $\mathbf{C}\mathcal{C}$, its objects are the $n$-cells of $\mathcal{C}$ and its morphisms from $f$ to $g$ are the contexts $C$ of $\mathcal{C}$ such that $C[f] = g$ holds. We denote by $\mathbf{W}\mathcal{C}$ the subcategory of $\mathbf{C}\mathcal{C}$ with the same objects and with whiskers as morphisms.

**1.1.7. *Natural systems.*** — A *natural system on* $\mathcal{C}$ is a functor $D$ from $\mathbf{C}\mathcal{C}$ to the category of groups. We denote by $D_u$ and $D_C$ the images of an $n$-cell $u$ and of a context $C$ of $\mathcal{C}$ by the functor $D$. When no confusion arise, we write $C[a]$ instead of $D_C(a)$. A natural system $D$ on $\mathcal{C}$ is *abelian* when $D_u$ is an abelian group for every $n$-cell $u$.

**1.2. Rewriting properties of polygraphs.** — We fix an $(n+1)$-polygraph $\Sigma$ throughout this section.

**1.2.1. *Termination.*** — One says that an $n$-cell $u$ of $\Sigma_n^*$ *reduces* into an $n$-cell $v$ when $\Sigma^*$ contains a non-identity $(n+1)$-cell with source $u$ and target $v$. One says that $u$ is a *normal form* when it does not reduce into an $n$-cell. A *normal form of* $u$ is an $n$-cell $v$ which is a normal form and such that $u$ reduces into $v$. A *reduction sequence* is a countable family $(u_n)_{n \in I}$ of $n$-cells such that each $u_n$ reduces into $u_{n+1}$; it is *finite* or *infinite* when the indexing set $I$ is.