Séminaires & Congrès

COLLECTION SMF

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OPERADS 2009

Numéro 26

Jean-Louis Loday & Bruno Vallette, ed.

SOCIÉTE MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
26, 2011, p. 163–179

DIAGRAM REWRITING AND OPERADS

by

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Abstract. — We give a survey of a diagrammatic syntax for *PROs* and *PROPs*, which are related to the theory of operads and bialgebras. Using *diagram rewriting*, we obtain *presentations of PROs by generators and relations.* In some cases, we even get *convergent rewrite systems.*

Résumé **(Réécriture de diagrammes et opérades).** — Nous donnons un aperçu de la syntaxe diagrammatique pour les *PROs* et les *PROPs*, qui sont liés à la théorie des opérades et des bigèbres. En utilisant la *réécriture de diagrammes*, on obtient des *présentations de PROs par générateurs et relations.* Dans certains cas, on obtient même des *systèmes de réécriture convergents.*

Except for Sections 4 and 7, most of the material presented in this paper comes from [12], which was inspired by [2].

1. PROs and PROPs

Definition 1. — *A* PRO *(or* product category*) is a* strict monoidal category*, that is a (small) category* $\mathbf{C}$ *equipped with some associative functor* $* : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *and a unit object, such that the set of objects of* $\mathbf{C}$ *is* $\mathbb{N}$*, and* $p * q = p + q$ *for all* $p, q \in \mathbb{N}$*. In particular, the unit object is* 0*.*

In order to define a PRO, since objects are already known, it suffices to give the set $\mathbf{C}(p, q)$ of *morphisms* $f : p \to q$ for all $p, q \in \mathbb{N}$, together with:

- a *sequential composition* $g \circ f : p \to r$ for any $f : p \to q$ and $g : q \to r$;
- a *parallel composition* $f * f' : p + p' \to q + q'$ for any $f : p \to q$ and $f' : p' \to q'$;
- an *identity* $\mathrm{id}_p : p \to p$ for all $p \in \mathbb{N}$.

2000 Mathematics Subject Classification. — 18C10.
Key words and phrases. — algebraic theory, braid, canonical form, confluence, diagram rewriting, Knuth-Bendix completion, operad, permutation, presentation by generators and relations, PRO, PROP, Schur functor, tangle.

This terminology will be clear in the next section. Of course, those two compositions must be associative, with units:

- $(h \circ g) \circ f = h \circ (g \circ f)$ for any $f : p \to q$, $g : q \to r$, and $h : r \to s$;
- $(f * f') * f'' = f * (f' * f'')$ for any $f : p \to q$, $f' : p' \to q'$, and $f'' : p'' \to q''$;
- $f \circ \mathrm{id}_p = f = \mathrm{id}_q \circ f$ and $f * \mathrm{id}_0 = f = \mathrm{id}_0 * f$ for any $f : p \to q$.

But they must also be compatible (law of *interchange*):

- $(g \circ f) * (g' \circ f') = (g * g') \circ (f * f')$ for any $f : p \to q$, $g : q \to r$, $f' : p' \to q'$, and $g' : q' \to r'$;
- $\mathrm{id}_p * \mathrm{id}_q = \mathrm{id}_{p+q}$ for all $p, q \in \mathbb{N}$.

Here are typical examples:

- the PRO $\mathfrak{F}$, where a morphism $f : p \to q$ is a map from $\{1, \ldots, p\}$ to $\{1, \ldots, q\}$;
- the PRO $\mathfrak{M} \subset \mathfrak{F}$, where a morphism $f : p \to q$ is a monotone map from $\{1, \ldots, p\}$ to $\{1, \ldots, q\}$;
- the PRO $\mathbf{L}(\mathbb{K})$, where a morphism $f : p \to q$ is a $\mathbb{K}$-linear map from $\mathbb{K}^p$ to $\mathbb{K}^q$ (or a $q \times p$ matrix) for any commutative field $\mathbb{K}$.

Compositions are obvious:

- $\circ$ is composition of maps (or product of matrices);
- $*$ is disjoint union (for $\mathfrak{F}$), ordered sum (for $\mathfrak{M}$), or direct sum (for $\mathbf{L}(\mathbb{K})$).

If we remove the object 0 from the PRO $\mathfrak{M}$, then we get the *simplicial category* Δ.

Definition 2. — *A PRO* $\mathbf{C}$ *is* reversible *if all* $\mathbf{C}(p,p)$ *are groups and* $\mathbf{C}(p,q) = \varnothing$ *whenever* $p \neq q$.

In order to define such a PRO, it suffices to give a group $\mathbf{C}_p = \mathbf{C}(p,p)$ for all p, together with a parallel composition $f * g \in \mathbf{C}_{p+q}$ defined for any $f \in \mathbf{C}_p$ and $g \in \mathbf{C}_q$. Note that a reversible PRO is a groupoid, but the condition $\mathbf{C}(p,q) = \varnothing$ for $p \neq q$ is not necessary to get a groupoid. Here are typical examples:

- the PRO $\mathfrak{S} \subset \mathfrak{F}$, where $\mathfrak{S}_p$ is the p-th *symmetric group*;
- the PRO $\mathfrak{B}$, where $\mathfrak{B}_p$ is the p-th *braid group*;
- the PRO $\mathbf{GL}(\mathbb{K}) \subset \mathbf{L}(\mathbb{K})$, where $\mathbf{GL}_p(\mathbb{K})$ is the p-th *linear group over* $\mathbb{K}$;
- the PRO $\mathbf{O} \subset \mathbf{GL}(\mathbb{R})$, where $\mathbf{O}_p \subset \mathbf{GL}_p(\mathbb{R})$ is the p-th *orthogonal group*.

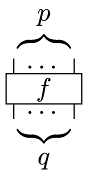
Definition 3. — *A* PROP *(or* product and permutation category*) is a PRO* $\mathbf{C} \supset \mathfrak{S}$.

For instance, both $\mathfrak{F}$ and $\mathbf{L}(\mathbb{K})$ are PROPs, but not $\mathfrak{M}$. PROPs are introduced in [16], with a slightly different definition, but of course, our notion is equivalent.

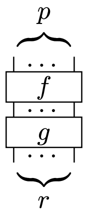
2. Diagrams

We recall the *diagrammatic syntax* of [12]:

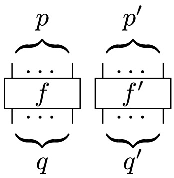
– a morphism $f : p \to q$ is pictured as a box with p inputs and q outputs:



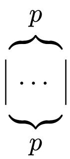
– for $f : p \to q$ and $g : q \to r$, the sequential composition $g \circ f : p \to r$ is pictured as follows:



– for $f : p \to q$ and $f' : p' \to q'$, the parallel composition $f * f' : p + p' \to q + q'$ is pictured as follows:



– the identity $\mathrm{id}_p : p \to p$ is pictured as follows:



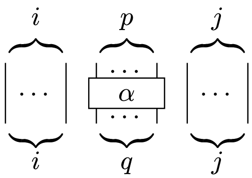
– in particular, $\mathrm{id}_0 : 0 \to 0$ is pictured as an empty diagram.

Definition 4. — *A* signature *is a graph $\mathcal{S}$ with vertices in $\mathbb{N}$. An edge $\alpha : p \to q$ in $\mathcal{S}$ is called a* symbol with p inputs and q outputs.

For instance, the following signature will be introduced in the next section:

$$\begin{tikzcd} 0 \arrow[r, "\eta"] & 1 & 2 \arrow[l, "\mu"'] \arrow[loop right, "\tau"] \end{tikzcd}$$

Definition 5. — *An* elementary diagram *built over signature $\mathcal{S}$ is a formal parallel composition* $\mathrm{id}_i * \alpha * \mathrm{id}_j : i+p+j \to i+q+j$*, where* $\alpha : p \to q$ *is a symbol of* $\mathcal{S}$*, and* $i, j \in \mathbb{N}$*. It is pictured as follows:*

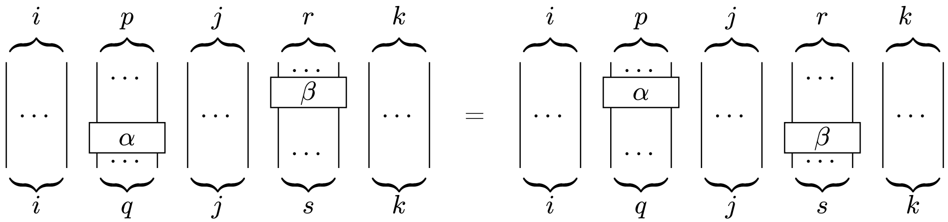


Definition 6. — *A* diagram *built over signature $\mathcal{S}$ is a formal sequential composition* $\phi_n \circ \cdots \circ \phi_1 : p_0 \to p_n$*, where* $\phi_1 : p_0 \to p_1, \phi_2 : p_1 \to p_2, \ldots, \phi_n : p_{n-1} \to p_n$ *are elementary diagrams. In particular, we get* $\mathrm{id}_{p_0} : p_0 \to p_0$ *when* $n = 0$*.*

Definition 7. — *The* free PRO $\mathcal{S}^*$ *consists of all diagrams built over a signature* $\mathcal{S}$*, modulo the* commutation laws*:*

$$(\mathrm{id}_i * \alpha * \mathrm{id}_{j+s+k}) \circ (\mathrm{id}_{i+p+j} * \beta * \mathrm{id}_k) = (\mathrm{id}_{i+q+j} * \beta * \mathrm{id}_k) \circ (\mathrm{id}_i * \alpha * \mathrm{id}_{j+r+k})$$

for any symbols $\alpha : p \to q$ *and* $\beta : r \to s$*, and for all* $i, j, k \in \mathbb{N}$*.*



The commutation laws are necessary to get a PRO, which must satisfy interchange. In fact, a morphism of $\mathcal{S}^*$ can also be considered as a formal (sequential and parallel) composition of symbols modulo associativity, units, and interchange.

3. Presentations by generators and relations

Definition 8. — *A* relation $\rho = \sigma$ *(over a signature* $\mathcal{S}$*) is given by two diagrams* $\rho, \sigma : p \to q$ *built over* $\mathcal{S}$*.*

Definition 9. — *A* presentation *of a PRO* $\mathbf{C}$ *consists of a signature* $\mathcal{S}$ *together with a set* $\mathcal{R}$ *of relations over* $\mathcal{S}$*, such that* $\mathbf{C} \simeq \mathcal{S}^* / \leftrightarrow^*_{\mathcal{R}}$*, where* $\leftrightarrow^*_{\mathcal{R}}$ *is the* congruence generated by $\mathcal{R}$*.*