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Frédéric Chapoton

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CATEGORIFICATION OF THE DENDRIFORM OPERAD

by

Frédéric Chapoton

Abstract. — The idea of "categorification" of an algebraic structure has already been applied successfully by many authors to algebras or Hopf algebras. We propose here to apply this idea to the more sophisticated algebraic notion of operad: we try to categorify the dendriform operad, which was introduced by Loday. The underlying Abelian groups of the dendriform operad are categorified by the Abelian categories of modules over the incidence algebras of some posets, usually called the Tamari posets. The linear maps that make the structure of the dendriform operad should then be categorified by functors between these Abelian categories.

We make here a first step in this direction by describing a functor that categorifies the first composition map of the dendriform operad. We also obtain functors that categorify the other composition maps of the dendriform operad, but only at the prize of passing to the derived categories of the categories of modules, which is probably avoidable.

***Résumé* (Catégorification de l'opérade dendriforme).** — L'idée de « catégorification » d'une structure algébrique a déjà été appliquée avec succès par de nombreux auteurs à des algèbres associatives ou des algèbres de Hopf. On se propose ici de l'appliquer à une structure algébrique plus complexe, celle d'opérade. On tente plus spécifiquement de catégorifier l'opérade dendriforme introduite par Loday. Les groupes abéliens sous-jacents à l'opérade dendriforme sont catégorifiés par les catégories abéliennes de modules sur les algèbres d'incidence de certains ensembles partiellement ordonnés, les posets de Tamari. Les applications linéaires qui forment la structure d'opérade doivent être catégorifiées par des foncteurs entre ces catégories abéliennes.

On franchit ici un première étape dans cette direction, en décrivant un foncteur qui catégorifie la première application de composition de l'opérade dendriforme. On obtient aussi des foncteurs qui catégorifient les autres applications de composition, mais seulement au prix d'un passage, sans doute superflu, aux catégories dérivées des catégories de modules.

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1. Introduction

The idea of "categorification" is the very general idea that one can gain understanding, in many specific situations, by trying to replace *sets* by *categories*, elements of sets by objects in categories and maps between sets by functors, in such a way that one recovers the initial set as the *set of isomorphism classes* of objects of the category. For example, one can fruitfully replace the set of positive integers by the category of finite sets. As there is some freedom in the choice of morphisms in the category of finite sets, "categorification" is by no way unique. This idea has for instance been used in combinatorics by the theory of species [13].

There is a variation or refinement of this idea, in which one should instead try to replace *Abelian groups* by *Abelian categories*, elements of Abelian groups by objects of Abelian categories and linear maps by additive functors, in such a way that one recovers the initial Abelian group as the *Grothendieck group* of the Abelian category. For example, the Abelian group $\mathbb{Z}$ is categorified in this sense by the category of vector spaces over a field. It is this "additive" kind of categorification that we will have in mind from now on.

Categorification has already been applied with great success in very different contexts. To name just a few, one can cite the work of Khovanov on the categorification of the Jones polynomial in knot theory [14], the relation of quasi-symmetric functions with modules over 0-Hecke algebras by Krob and Thibon [9, 15] or the recent work by Hernandez and Leclerc on the "monoidal" categorification of cluster algebras [11]. Closely related is also the general tendency in algebraic geometry to use (derived) categories of coherent sheaves rather than K-theory groups.

In this article, we want to use the idea of categorification for an operad. Indeed, a (non-symmetric) operad $\mathcal{P}$ is just a collection $\mathcal{P}(n)$ of Abelian groups for each integer $n \geq 1$ and bilinear maps between them, with some appropriate axioms. One has therefore to define appropriate Abelian categories and bifunctors between them, in such a way that the Grothendieck group construction recovers the initial operad.

The operad we want to study here is the dendriform operad Dend. It was introduced in the 1990's by Loday, together with several other operads [18, 19]. The dendriform operad has been studied a lot since its definition. Loday himself has shown that the operad Dend is Koszul. Later, it was proved in [5] that the operad Dend is anticyclic.

For any operad $\mathcal{P}$, there is an associated notion of algebra over $\mathcal{P}$. In the case of the operad Dend, this is called a dendriform algebra. This notion can be described as an associative algebra, together with a decomposition of the associative product as the sum of two binary products, with some appropriate axioms. Loday has proved that the free dendriform algebras can be described using classical combinatorial objects called planar binary trees. This also provides the description of the underlying Abelian groups of the dendriform operad.

Moreover, there has been several hints that the dendriform algebras and operad are closely related to a natural partial order on planar binary trees, called the Tamari poset. First, Loday and Ronco have described the associative product $*$ on free dendriform algebras, in their basis indexed by planar binary trees: the product of two planar binary trees is the sum over an interval in a Tamari poset [22]. Next, the anticyclic structure of the operad Dend is given by a collection of linear maps that can be described directly from the Tamari poset [8].

As the Grothendieck groups of the Abelian categories of modules over the incidence algebras of the Tamari posets have a basis indexed by planar binary trees, it is natural enough to make a guess that these categories may serve as a categorification of the underlying Abelian groups of the dendriform operad, spanned by planar binary trees.

There remains to define functors that categorify the linear maps that belong to the dendriform operad. More precisely, there is a composition map

$$\circ_i : \mathrm{Dend}(m) \otimes_{\mathbb{Z}} \mathrm{Dend}(n) \to \mathrm{Dend}(m+n-1)$$

for each i between 1 and m.

In this article, we define functors $\circ_1$ that categorify the $\circ_1$ composition maps. The situation concerning the other composition maps is not satisfactory: we can define functors $\circ_i$ for $i > 1$, but only as functors between derived categories of the category of modules. This is not an explicit construction, and it should be replaced by a better one, working for the categories of modules.

We have also obtained functors corresponding to the following bilinear maps, which are not part of the operad structure, but still closely related to the situation:

- the associative product $* : \mathrm{Dend}(m) \otimes_{\mathbb{Z}} \mathrm{Dend}(n) \to \mathrm{Dend}(m+n)$,
- a new associative product $\# : \mathrm{Dend}(m) \otimes_{\mathbb{Z}} \mathrm{Dend}(n) \to \mathrm{Dend}(m+n-1)$.

The product $*$ is the associative product of the free dendriform algebra on one generator. The product # has been explained to the author by Jean-Christophe Aval and Xavier Viennot, in the setting of Catalan alternative tableaux, see [3] and [24] for related works.

One important tool in this article is a small set-operad contained in Dend, introduced in [7]. This sub-operad can be described using noncrossing configurations (called noncrossing plants and noncrossing trees) in a regular polygon.

The article is organised as follows. In section 2, we recall the necessary material on the combinatorics of planar binary trees, the Tamari poset, the dendriform operad and dendriform algebras, and representation theory. In section 3, we recall the combinatorics of noncrossing trees and noncrossing plants, together with the operad structure on these objects. Section 4 is the technical core of the article, where we obtain several descriptions of the set of projective noncrossing trees. In section 5, we apply the results of the previous section to define functors that categorify the composition maps of the dendriform operad. In section 6, we categorify the associative product $*$ and the associative product #.

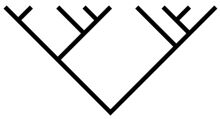

FIGURE 1. A planar binary tree of degree 8

2. General setup

2.1. Planar binary trees. — Let us first introduce very classical combinatorial objects, called **planar binary trees**. They can be concisely defined as follows: a planar binary tree is either the trivial tree $|$ or a pair of planar binary trees (x, y).

We will always draw planar binary trees with their leaves at the top, on a horizontal line, and their root at the bottom, as in Figure 1.

Let us define the **degree** of a planar binary tree to be the number of leaves minus one. Let $\mathbb{Y}_n$ be the set of planar binary trees of degree n. For example, $\mathbb{Y}_1 = \{\vee\}$ and $\mathbb{Y}_2 = \{((|,|),|), (|,(|,|))\}$. The cardinal of $\mathbb{Y}_n$ is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Let $\mathbb{Y}$ be the union of the sets $\mathbb{Y}_n$ for $n \geq 1$.

Let us now recall two basic combinatorial operations on $\mathbb{Y}$: the over product $/$ and the under product $\backslash$, both associative and graded.

Let x, y be planar binary trees. The planar binary tree x/y is obtained by identifying the root of x with the leftmost leaf of y. Similarly $x\backslash y$ is obtained by identifying the root of y with the rightmost leaf of x. For example, $\vee/\vee = ((|,|),|)$ and $\vee\backslash\vee = (|,(|,|))$. Note that one has $(x/y)\backslash z = x/(y\backslash z)$.

Lemma 2.1. — *Any planar binary tree of degree at least 2 can either be written x/y for some planar binary trees x, y or $\vee\backslash x$ for some planar binary tree x.*

Proof. — If there is at least one vertex to the left of the root, then the first decomposition is possible. Else one must be in the second case. □

There is an obvious involution on planar binary trees, the (left-right) reversal, that will be denoted $x \mapsto \overline{x}$, exchanging the over and under products: $\overline{x/y} = \overline{y}\backslash\overline{x}$

2.2. Tamari poset. — There is a partial order on the set $\mathbb{Y}_n$, called the Tamari poset. It was introduced by Dov Tamari in [12] and proved there to be a lattice.

This partial order can be defined as the transitive closure of **elementary moves**: $x \leq y$ if x is obtained from y by a sequence of local changes, replacing the configuration $(|,(|,|))$ by the configuration $((|,|),|)$ somewhere in the tree.

The elementary moves can be described more formally as follows:

- For any planar binary trees a, b, c, there is an elementary move from $a/\vee\backslash(b/\vee\backslash c)$ to $(a/\vee\backslash b)/\vee\backslash c$.
- If $x \to y$ is an elementary move, then $x/z \to y/z$, $z/x \to z/y$ and $\vee\backslash x \to \vee\backslash y$ are also elementary moves.