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FREENESS THEOREMS FOR OPERADS VIA GRÖBNER BASES

by

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Abstract. — We show how to use Gröbner bases for operads to prove various freeness theorems: freeness of certain operads as nonsymmetric operads, freeness of an operad $\mathscr{Q}$ as a $\mathscr{P}$-module for an inclusion $\mathscr{P} \hookrightarrow \mathscr{Q}$, freeness of a suboperad. This gives new proofs of many known results of this type and helps to prove some new results.

***Résumé* (Théorèmes sur la liberté de certaines opérades par les bases de Gröbner)**
Nous montrons comment utiliser les bases de Gröbner des opérades pour démontrer que certaines d'entre elles sont libres : libre en tant qu'opérade non-symétrique, opérade $\mathscr{Q}$ libre comme $\mathscr{P}$-module pour une inclusion $\mathscr{P} \hookrightarrow \mathscr{Q}$ et opérade libre comme sous-opérade. Ceci fournit de nouvelles démonstrations pour des résultats déjà connus et permet d'en démontrer de nouveaux.

1. Introduction

1.1. Description of results. — Recently, many freeness theorems about operads and free algebras over various operads have been proved. An incomplete list includes the following results:

- free dendriform algebras are free as associative algebras (Loday and Ronco [12]);
- free pre-Lie algebras are free as Lie algebras (Chapoton [4] and Foissy [8]);
- free algebras with two compatible associative products are free as associative algebras (the author's result [6]);
- the nonsymmetric operads Lie and PreLie are free (Salvatore and Tauraso [15], Bergeron and Livernet [2]);
- the suboperad of the operad PreLie generated by the symmetrized pre-Lie product is free (Bergeron and Loday [3]).

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In this article, we apply Gröbner bases for operads to derive several freeness theorems that imply all these results and several new ones. Our freeness theorems remind of the Magnus's Freiheitssatz from the group theory [13] and its analogues in other branches of algebra. Many of the results of this paper can be obtained by direct computations that use the Gröbner basis algorithm [7], however, for we tried to replace most of computations by some ideas coming from the Koszul duality theory [10].

1.2. Outline of the paper. — The paper is organized as follows. In Section 2, we briefly recall shuffle operads and Gröbner bases. In Section 3, we give a general criterion for a symmetric operad to be free as a nonsymmetric operad, and show how this criterion applies to the cases of Lie, PreLie, and Lie^2. In Section 4, we give a criterion for a mapping of operads to be an embedding, and apply it to deduce the inclusion $\mathrm{Mag} \hookrightarrow \mathrm{PreLie}$. In Section 5, we give a criterion of freeness as a module, and show how this criterion applies to pairs $(\mathrm{Lie}, \mathrm{PreLie})$ and, for a certain class of quadratic operads, $(\mathscr{O}, \mathscr{O}^2)$ where $\mathscr{O}^2$ denotes the operad of (weakly) compatible $\mathscr{O}$-structures.

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2. Shuffle operads and Gröbner bases

All vector spaces throughout this work are defined over an arbitary field k of zero characteristic.

In this section, we give, mostly following [7], a brief outline of definitions and the most important facts. For details on symmetric operads and Koszul duality, see [10] and [14]. For more details on shuffle operads and Gröbner bases, see [7].

2.1. Shuffle compositions. — We denote by Ord the category of nonempty finite ordered sets (with order-preserving bijections as morphisms), and by Fin — the category of nonempty finite sets (with bijections as morphisms). Also, we denote by Vect the category of vector spaces (with linear operators as morphisms; unlike the first two cases, we do not require a map to be invertible).

Definition 1. — 1. A *(nonsymmetric) collection* is a functor from the category Ord to the category Vect.

2. A *symmetric collection* (or an $\mathbb{S}$-*module*) is a functor from the category Fin to the category Vect.

For either type of collections, we can consider the category whose objects are collections of this type (and morphisms are morphisms of the corresponding functors). The natural forgetful functor ${}^f\colon \mathrm{Ord} \to \mathrm{Fin}$, $I \mapsto I^f$ leads to a forgetful functor ^{f} from the category of symmetric collections to the category of nonsymmetric ones, $\mathscr{P}^f(I) := \mathscr{P}(I^f)$.

The following monoidal structures on our categories are important for the theory of operads.

Definition 2. — Let $\mathscr{P}$ and $\mathscr{Q}$ be two nonsymmetric collections. Define their *(nonsymmetric) composition* $\mathscr{P} \circ \mathscr{Q}$ by the formula

$$(\mathscr{P} \circ \mathscr{Q})(I) := \bigoplus_k \mathscr{P}(k) \otimes \left(\bigoplus_{f\colon I \twoheadrightarrow [k]} \mathscr{Q}(f^{-1}(1)) \otimes \cdots \otimes \mathscr{Q}(f^{-1}(k)) \right),$$

where the sum is taken over all non-decreasing surjections f.

Let $\mathscr{P}$ and $\mathscr{Q}$ be two nonsymmetric collections. Define their *shuffle composition* $\mathscr{P} \circ_{sh} \mathscr{Q}$ by the formula

$$(\mathscr{P} \circ_{sh} \mathscr{Q})(I) := \bigoplus_k \mathscr{P}(k) \otimes \left(\bigoplus_{f\colon I \twoheadrightarrow [k]} \mathscr{Q}(f^{-1}(1)) \otimes \cdots \otimes \mathscr{Q}(f^{-1}(k)) \right),$$

where the sum is taken over all shuffling surjections f, that is surjections for which $\min f^{-1}(i) < \min f^{-1}(j)$ whenever $i < j$.

Let $\mathscr{P}$ and $\mathscr{Q}$ be two symmetric collections. Define their *(symmetric) composition* $\mathscr{P} \circ \mathscr{Q}$ by the formula

$$(\mathscr{P} \circ \mathscr{Q})(I) := \bigoplus_k \mathscr{P}(k) \otimes_{\Bbbk S_k} \left(\bigoplus_{f\colon I \twoheadrightarrow [k]} \mathscr{Q}(f^{-1}(1)) \otimes \cdots \otimes \mathscr{Q}(f^{-1}(k)) \right),$$

where the sum is taken over all surjections f.

Definition 3. — 1. A *nonsymmetric operad* is a monoid in the category of nonsymmetric collections with the monoidal structure given by the nonsymmetric composition.
2. A *shuffle operad* is a monoid in the category of nonsymmetric collections with the monoidal structure given by the shuffle composition.
3. A *symmetric operad* is a monoid in the category of symmetric collections with the monoidal structure given by the (symmetric) composition.

It turns out that the forgetful functor is a monoidal functor between the category of symmetric operads and the category of shuffle operads. Consequently, it turns out that to study various questions of linear algebra for operads, it is sufficient to forget the full symmetric structure because the shuffle structure already captures everything. Further in this section, the word "operad" means a shuffle operad.

2.2. Tree monomials, divisibility, and Gröbner bases. — We use the usual way to represent operadic elements by decorated rooted trees. A tree has (internal) vertices, directed edges, and inputs (leaves). For a tree whose leaves are labelled by an ordered set, its canonical planar representative is defined as follows. In general, an embedding of a (rooted) tree in the plane is determined by an ordering of inputs for each vertex. To compare two inputs of a vertex v, we find the minimal leaves that one can reach from v via the corresponding inputs. The input for which the minimal leaf is smaller is considered to be less than the other one. Note that this choice of a representative is essentially the same one as we already made when we identified symmetric compositions with shuffle compositions.

Let us introduce an explicit realisation of the free operad generated by a collection $\mathscr{V}$. The basis of this operad will be indexed by planar representative of trees with decorations of all vertices. First of all, the simplest possible tree is the degenerate tree (with no internal vertices); it corresponds to the unit of our operad. The second simplest type of trees is given by corollas, that is trees with one vertex. We shall fix a basis $B^{\mathscr{V}}$ of $\mathscr{V}$ and decorate the vertex of each corolla with a basis element; for a corolla with n inputs, the corresponding element should belong to the basis of $\mathscr{V}(n)$. The basis for whole free operad consists of all planar representatives of trees built from these corollas (explicitly, one starts with this collection of corollas, defines compositions of trees in terms of grafting, and then considers all trees obtained from corollas by iterated shuffle compositions). We shall refer to elements of this basis as *tree monomials*.

An ordering of tree monomials of $\mathscr{F}_{\mathscr{V}}$ is said to be *admissible*, if it is compatible with the operadic structure, that is, replacing the operations in any shuffle compositions with larger operations of the same arities increases the result of the composition. Here we shall describe several admissible orderings which suit our purposes. All results of this section are valid for every admissible ordering of tree monomials.

Recall the following construction crucial for the "path-lexicographic ordering" [7]. Let α be a tree monomial with n inputs. We associate to α a sequence $(a_1, a_2, \ldots, a_n)$ of n words in the alphabet $B^{\mathscr{V}}$ and a permutation $g \in S_n$ as follows. For each leaf i of the underlying tree τ, there exists a unique path from the root to i. The word a_i is the word composed, from left to right, of the labels of the vertices of this path, starting from the root vertex. The permutation g lists the labels of leaves of the underlying tree in the order determined by the planar structure (from left to right).

To compare two tree monomials we always compare their arities first. If the arities are equal, there are several different options of how to proceed. Recall that an ordering of words in the alphabet $B^{\mathscr{V}}$ is said to be admissible, if it is compatible with the