Séminaires & Congrès

COLLECTION SMF

PEN PROBLEMS IN THE THEORY OF AMPL FIELDS

Lior Bary-Soroker & Arno Fehm

THÉORIES DE GALOIS GÉOMÉTRIQUE ET DIFFÉRENTIELLE

Numéro 27 D. Bertrand, P. Boalch, J.-M. Couveignes, P. Dèbes, éds.

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
27, 2012, p. 1–11

OPEN PROBLEMS IN THE THEORY OF AMPLE FIELDS

by

Lior Bary-Soroker & Arno Fehm

Abstract. — Fifteen years after their discovery, ample fields now stand at the center of research in contemporary Galois theory and attract more and more attention also from other areas of mathematics. This survey gives an introduction to the theory of ample fields and discusses open problems.

***Résumé* (Problèmes ouverts de la théorie des corps amples).** — Quinze ans après leur découverte, les corps amples se situent maintenant au cœur de la recherche contemporaine en théorie de Galois et attirent de plus en plus l'attention de la part d'autres branches des mathématiques. Ce travail présente une introduction à la théorie des corps amples et traite de quelques problèmes ouverts.

1. Introduction

In the middle of the 1990's, Pop realized that all fields for which a certain Galois theoretic conjecture was proven (namely the regular solvability of finite split embedding problems) share a common property: the set of rational points of any smooth curve over such a field is either empty of infinite. Moreover, in [**Pop96**] he showed that the conjecture actually holds for all fields satisfying this property. Since then these fields, nowadays called *ample* fields[(1)] – the topic of this survey – play a central role in Galois theory.

There are several equivalent ways to define ample fields, and this notion captures well the intuitive concept of a 'large' field. For example, the class of ample fields subsumes several seemingly unrelated classes of large fields, like algebraically closed, real closed, separably closed, and Henselian valued fields. A typical example of an ample field that does not fall into any of these categories is the field of totally S-adic numbers $\mathbb{Q}_{\mathrm{tot},S}$, S a finite set of places of $\mathbb{Q}$ – the maximal extension of $\mathbb{Q}$ in which each place in S is totally split. For example, the field of totally real algebraic numbers, which plays an important role in number theory, is of this kind.

(1) Some authors prefer the term *large* field.

In recent years, ample fields attracted more and more attention also in other subjects, like arithmetic geometry [**Dèb98**, **Kol99**, **MB01**, **FP10**], valuation theory [**Kuh04**, **AKP11**], and model theory [**Koe02**, **Tre05**, **JK10**].

In this note we survey the basics of the theory of ample fields and discuss open problems. The Galois-theoretic aspects of the theory of ample fields are covered well in the literature, and we refer the reader to the beautiful survey paper [**DD99**] of Dèbes and Deschamps, and the comprehensive book [**Jar11**] by Jarden. We will focus on new developments that took place since the publication of [**DD99**], and on connections between the theory of ample fields and other subjects.

2. Background

2.1. Characterization. — Apart from being given different names (like large, anti-Mordellic, fertile, pop) there are also several equivalent ways to define the class of ample fields, cf. [**Pop96**]:

Proposition 2.1. — *The following properties for a field K are equivalent:*

(1) *Every smooth K-curve with a K-rational point has infinitely many such points.*
(2) *If V is a smooth K-variety, then the set of K-rational points of V is either empty or Zariski-dense in V.*
(3) *K is existentially closed in the field of formal Laurent series $K((t))$.*
(4) *K is existentially closed in some field extension that admits a nontrivial Henselian valuation.*

Here, K is called *existentially closed* in an extension F if every existential first-order sentence in the language of rings with parameters from K which holds in F, also holds in K. Equivalently, K is existentially closed in F if for any $x_1, \dots, x_n \in F$ there exists a K-homomorphism from the ring $K[x_1, \dots, x_n]$ to K. An **ample** field is a field K that satisfies the equivalent conditions (1)-(4).

2.2. Ample and non-ample fields. — The most important known properties of fields that imply ampleness can be grouped into three classes:

1. arithmetic-geometric
2. topological
3. Galois-theoretic

Arithmetic-geometric properties: Every algebraically closed, separably closed, and more generally, pseudo algebraically closed field, is ample. Here, a field K is called pseudo algebraically closed (PAC, see [**FJ08**]) if every absolutely irreducible K-variety has a K-rational point. Even more generally, fields that satisfy a geometric local-global principle for varieties are ample, [**Pop96**]. An example of such fields are the PSC fields: If S is a finite set of places of $\mathbb{Q}$, then a field $K \subseteq \mathbb{Q}_{\mathrm{tot},S}$ is called PSC if every smooth $\mathbb{Q}$-variety that has a point over $\mathbb{Q}_p$ for each $p \in S$ (with $\mathbb{Q}_\infty = \mathbb{R}$) has a K-rational point.

Topological properties: Every field which is complete with respect to a nontrivial absolute value is ample. Also, every field which admits a nontrivial Henselian valuation is ample. This can be generalized further to the quotient fields of domains that are complete (more generally Henselian) with respect to an ideal [**Pop10**], or a norm [**FP11**].

Galois-theoretic properties that imply ampleness are discussed in Section 4.

On the other side, there are three basic classes of non-ample fields:

1. finite fields
2. number fields, i.e. finite extensions of $\mathbb{Q}$
3. function fields, i.e. fields that are a finitely generated and transcendental extension of another field

In particular, all global fields are non-ample, in contrast to the fact that all local fields are ample. Apart from these three classes, only very few non-ample fields are known. As Dèbes puts it in [**Dèb98**]:

> "[...] it happens to be difficult to produce non-ample fields at all (at least inside $\bar{\mathbb{Q}}$ and apart from number fields)".

For some examples of non-ample fields that do not fall into any of the above three classes see [**Koe04**, **LR08**, **Feh11**]. Surprisingly, to the best of our knowledge, all infinite non-ample fields appearing in the literature are Hilbertian.

2.3. Properties. — As explained before, the notion of ample fields captures very well the intuitive concept of a large field. One of the properties every notion of large fields should certainly satisfy is the following, cf. [**Pop96**]:

Proposition 2.2. — *The class of ample fields is closed under algebraic extensions.*

In addition we have:

Proposition 2.3. — *The class of ample fields is an elementary class. In particular, it is closed under elementary equivalence in the language of rings.*

2.4. Abelian varieties. — Let A be an abelian variety defined over a finitely generated field K. It is known that the Mordell-Weil group $A(K)$ is finitely generated, and in particular

$$\mathrm{rank}(A(K)) := \dim_{\mathbb{Q}}(A(K) \otimes \mathbb{Q}),$$

the **rank** of A over K, is finite. On the other hand, if K is algebraically closed and not algebraic over a finite field, then $\mathrm{rank}(A(K)) = \infty$. At least in characteristic zero, this holds more generally for arbitrary ample fields, cf. [**FP10**]:

Proposition 2.4. — *Let K be an ample field of characteristic zero and A/K a non-zero abelian variety. Then* $\mathrm{rank}(A(K)) = \infty$.

Although one can construct a non-ample field over which every abelian variety has infinite rank (i.e. the converse of Proposition 2.4 does not hold), we do not know any natural example of this kind, and none which is algebraic over $\mathbb{Q}$.

3. Algebraic fields

We denote by $\mathbb{Q}_{\mathrm{ab}}$ the maximal abelian extension of $\mathbb{Q}$, which, by the Kronecker-Weber theorem, coincides with the maximal cyclotomic extension $\mathbb{Q}(\zeta_n : n \in \mathbb{N})$ of $\mathbb{Q}$.

3.1. — Our first question can be simply stated as follows.

Question I. — *Is $\mathbb{Q}_{\mathrm{ab}}$ ample?*

The importance of this question lies in the fact that, via Pop's results on ample fields, a positive answer would give a proof of the following conjecture of Shafarevich, cf. [DD99]:

***Conjecture 3.1* (Shafarevich)**. — *The absolute Galois group $G_{\mathbb{Q}_{\mathrm{ab}}}$ of $\mathbb{Q}_{\mathrm{ab}}$ is a free profinite group.*

As evidence for this conjecture one has Iwasawa's theorem that the maximal prosolvable quotient of $G_{\mathbb{Q}_{\mathrm{ab}}}$ is prosolvable free, and Tate's result that the cohomological dimension of $G_{\mathbb{Q}_{\mathrm{ab}}}$ is 1. Another piece of evidence is that the geometric Shafarevich conjecture – the analogue of the Shafarevich conjecture for global function fields – holds true, [Har95, Pop95]. A proof of Conjecture 3.1 could be seen as a step towards understanding the absolute Galois group of $\mathbb{Q}$, since $G_{\mathbb{Q}}$ is an extension of the well-known group $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^{\times}$ by $G_{\mathbb{Q}_{\mathrm{ab}}}$.

3.2. — It is interesting to note that it is even unknown whether the bigger field $\mathbb{Q}_{\mathrm{solv}}$, the maximal solvable Galois extension of $\mathbb{Q}$, is ample. Even worse, while Frey proved that $\mathbb{Q}_{\mathrm{ab}}$ is *not* PAC, [FJ08, Corollary 11.5.7], it is a long-standing open question whether $\mathbb{Q}_{\mathrm{solv}}$ is PAC, equivalently, whether every absolutely irreducible $\mathbb{Q}$-variety admits a solvable rational point.

The ampleness of $\mathbb{Q}_{\mathrm{tot},S}$ mentioned in the introduction follows from the fact that $\mathbb{Q}_{\mathrm{tot},S}$ is PSC, [MB89, GPR95]. This fact plays an important role in the study of potential modularity of Galois representations, which motivated Taylor to ask in [Tay04] whether also $\mathbb{Q}_{\mathrm{tot},S} \cap \mathbb{Q}_{\mathrm{solv}}$ is PSC. A positive answer to this would have "extremely important consequences", as he points out, and of course it would imply that $\mathbb{Q}_{\mathrm{tot},S} \cap \mathbb{Q}_{\mathrm{solv}}$ and consequently also $\mathbb{Q}_{\mathrm{solv}}$ are ample.

3.3. — A positive answer to Question I would also settle the following problem. In their 1974 paper [FJ74], Frey and Jarden investigate the rank of elliptic curves over certain fields which are not finitely generated. For example, they prove that any elliptic curve $E/\mathbb{Q}$ acquires infinite rank over the field $\mathbb{Q}(\sqrt{n} : n \in \mathbb{Z})$, so in particular $\mathrm{rank}(E(\mathbb{Q}_{\mathrm{ab}})) = \infty$. This led them to ask for a generalization of this result to arbitrary abelian varieties:

***Question 3.2* (Frey-Jarden)**. — *Is $\mathrm{rank}(A(\mathbb{Q}_{\mathrm{ab}})) = \infty$ for every non-zero abelian variety A over $\mathbb{Q}$?*