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FOLIATIONS ON THE MODULI SPACE OF RANK TWO CONNECTIONS ON THE PROJECTIVE LINE MINUS FOUR POINTS

by

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Abstract. — We look at natural foliations on the Painlevé VI moduli space of regular connections of rank 2 on $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}$. These foliations are fibrations, and are interpreted in terms of the nonabelian Hodge filtration, giving a proof of the nonabelian Hodge foliation conjecture in this case. Two basic kinds of fibrations arise: from apparent singularities, and from quasiparabolic bundles. We show that these are transverse. Okamoto's additional symmetry, which may be seen as Katz's middle convolution, exchanges the quasiparabolic and apparent-singularity foliations.

***Résumé* (Feuilletages de l'espace des modules des connexions de rang 2 sur la sphère privée quatre points)**

Nous considérons certains feuilletages naturels de l'espace des modules (de Painlevé VI) des connexions de rang 2 sur $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}$. Ces feuilletages sont des fibrations et s'interprètent en termes de filtration de Hodge non abélienne, donnant une preuve de la conjecture de feuilletage dans ce cadre. Essentiellement deux sortes de fibrations apparaissent : elles proviennent respectivement des singularités apparentes et de la structure sous-jacente de fibré quasiparabolique. Nous montrons qu'elles sont transverses. La symétrie d'Okamoto, qui peut-être vue comme la convolution moyenne de Katz, échange les deux types de feuilletages (singularité apparente et fibré quasiparabolique).

1. Introduction

The Painlevé VI equation is the isomonodromic deformation equation for systems of differential equations of rank 2 on $\mathbb{P}^1$ with four logarithmic singularities over $D :=$

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$\{t_1, t_2, t_3, t_4\}$. Such a system of differential equations is encoded in a vector bundle with logarithmic connection (E, ∇), where E is a vector bundle on $X = \mathbb{P}^1$ and $\nabla : E \to E \otimes \Omega^1_X(\log D)$ is a first order algebraic differential operator satisfying the Leibniz rule of a connection. At a singular point t_i the residue of ∇ is a linear endomorphism of E_{t_i}. The "space of initial conditions for Painlevé VI" is the moduli space of (E, ∇) such that the residues $\mathrm{res}(\nabla, t_i)$ lie in fixed conjugacy classes. The conjugacy class information is denoted $\mathbf{r}$, which for us will just mean fixing two distinct eigenvalues $r_i^{\pm}$ at each point. The isomonodromic evolution equation concerns what happens when the t_i move. However, in this paper we consider only the moduli space so the t_i are fixed.

The associated moduli stack is denoted by $\mathcal{M}^d(\mathbf{r})$. For generic choices of $\mathbf{r}$, all connections are irreducible and the moduli stack is a $\mathbb{G}_m$-gerb over the moduli space $M^d(\mathbf{r})$. Here d denotes the degree of the bundle E, related to $\mathbf{r}$ by the Fuchs relation (2.1). We usually assume that d is odd, essentially equivalent to $d = 1$, because any bundle of degree 1 having an irreducible connection must be of the form $B = \mathcal{O} \oplus \mathcal{O}(1)$. This facilitates the consideration of the parameter space for quasiparabolic structures.

The object of this paper is to study several natural fibrations on the moduli space. The second author, with Inaba and Iwasaki, have described the structure of $M^d(\mathbf{r})$ as obtained by several blow-ups of a ruled surface over $\mathbb{P}^1$ in [23, 24]. The function to $\mathbb{P}^1$ may be viewed as given by the position of an apparent singularity, considered also by Szabo [50] and Aidan [1]. The first author has considered this fibration too but also looked at the function from $M^d(\mathbf{r})$ to the space of quasiparabolic bundles [28], which as it turns out is again $\mathbb{P}^1$ or more precisely a non-separated scheme which had been introduced by Arinkin [2]. The third author has defined a decomposition of $M^d(\mathbf{r})$ obtained by looking at the limit of $(E, u\nabla)$ as $u \to 0$ into the moduli space of semistable parabolic Higgs bundles [47].

We compare these pictures by examining precisely the condition of stability depending on parabolic weight parameters. A choice of one of the two residues r_i^- is made at each point, and the eigenspace provides a 1-dimensional subspace $P_i \subset E_{t_i}$. The collection $(E, P_\bullet)$ is a quasiparabolic bundle [45]. Given that $E \cong B = \mathcal{O} \oplus \mathcal{O}(1)$, we can write down a parameter space for all quasiparabolic structures on B. The moduli stack for such quasiparabolic bundles is the stack quotient by $A = Aut(B)$.

Specifying two parabolic weights $\alpha_i^{\pm}$ at each point(*) transforms the quasiparabolic structures into parabolic ones for which there is a notion of stability. There is a collection of 8 inequalities concerning the parabolic weights appearing in Proposition 4.2: (a), (b) and 6 of type (c), see also (6.1) (6.2) (6.3). Depending on these inequalities,

(*) Here the smaller weight α_i^- is associated to the subspace P_i, which may be different from the convention used in some other papers.

generically the underlying parabolic bundle will either be stable, or unstable. The space of parabolic weights is therefore divided up into a stable zone, and 8 unstable zones.

The different unstable zones are permuted by the operation of performing two elementary transformations. Doing two at a time keeps the condition that the underlying bundle has odd degree. Up to these permutations, we can assume that we are in the (a)-unstable zone $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 < 1/2$ where $\epsilon_i = (\alpha_i^+ - \alpha_i^-)/2$. In this case, the subbundle $\mathcal{O}(1) \subset E = \mathcal{O} \oplus \mathcal{O}(1)$ is destabilizing. It determines an apparent singularity, which is the unique point at which the subbundle osculates to the direction of the connection. The position of this apparent singularity gives the map to $\mathbb{P}^1$. We point out in Theorem 5.6 that, in this unstable zone, this map is the same as the map taking (E, ∇) to the limiting α-stable Higgs bundle. This furnishes the comparison between the Higgs limit decomposition, and the fibration of [23, 24].

This comparison allows us to prove the foliation conjecture of [47] in this case. The Higgs limit decomposition is, from the definition, just a decomposition of the moduli space into disjoint locally closed subvarieties, which are Lagrangian for the natural symplectic structure. The foliation conjecture posits that this decomposition should be a foliation in the case when the moduli space is smooth. For the unstable zone, the decomposition is just the collection of connected components of the fibers of the smooth morphism of [23, 24] to $\mathbb{P}^1$, so it is a foliation.

We next turn our attention to the stable zone. The quasiparabolic bundles which support an irreducible connection with given residues are exactly the simple ones, and the quotient of the set of simple quasiparabolic structures by the automorphism group is the non-separated scheme $\mathcal{P}$ which is like $\mathbb{P}^1$ but has two copies of each t_i. This is the same as the space of leaves in the fibration corresponding to the unstable zone. It has also appeared in Arinkin's work [2] on the geometric Langlands program.

In the stable zone, the limit $\lim_{u\to 0}(E, u\nabla, P_\bullet)$ in the moduli space of α-stable parabolic Higgs bundles is just the underlying parabolic bundle $(E, P_\bullet)$, except at one from each pair of points lying over t_i. Thus, Theorem 6.2 says that in the stable zone, the Higgs limit decomposition is just the decomposition into fibers of the projection $M^d(\mathbf{r}) \to \mathcal{P}$ considered in [28], sending $(E, \nabla, P_\bullet)$ to $(E, P_\bullet)$. As before, this interpretation allows us to prove the foliation conjecture of [47] in this case.

Putting these together, we obtain a proof of the foliation conjecture for the moduli space of parabolic logarithmic connections of rank 2 on $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}$ with any generic residues and any generic parabolic weights. The genericity condition is nonresonance plus a natural condition which has been introduced by Kostov, ruling out the possibility of reducible connections. The combination of these two conditions will be called "nonspeciality".

In Section 7 we point out that this discussion gives the same results for the case of local systems on a root stack. These correspond to parabolic logarithmic connections on $\mathbb{P}^1$ whose residues and weights are the same rational numbers. In the root stack interpretation, the Higgs limit decomposition may be tied back to the same thing on a compact curve, a cyclic covering of $\mathbb{P}^1$ branched over t_1, t_2, t_3, t_4.

In Section 8 we show that the two different kinds of fibrations, obtained from apparent singularities and from the quasiparabolic structure, are strongly transverse: generic fibers intersect once. A similar picture has been described by Arinkin and Lysenko [3] when we switch to trace-free connections (and $\deg(E) = 0$).

In Section 9 we recall the additional Okamoto symmetry, and the fact pointed out by the first author in [28] that it interchanges the two different types of fibrations considered above. The geometrical picture was also investigated in [3]. Then in Section 10, we propose a possible explanation by interpreting Okamoto's additional symmetry as Katz middle convolution. This interpretation is now well known, apparently first pointed out by Boalch [7] [8] [9], Dettweiler and Reiter [16], and Crawley-Boevey [14].

We calculate, concentrating on the case of finite order monodromy, that a middle convolution with suitably chosen rank 1 local system interchanges the stable and unstable zones. Assuming a compatibility of higher direct images which is not yet proven, the middle convolution will preserve the Higgs limit decomposition and this property would imply that it permutes the two different kinds of foliations.

As a part of the numerous ongoing investigations of the rich structure of these moduli spaces, the present discussion points out the role of the different regions in the space of parabolic weights. Nevertheless, a number of further questions remain open in this direction, such as what happens along the hyperplanes of special values of residues and/or parabolic weights. We hope to address these in the future.

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2. Moduli stacks of parabolic logarithmic λ-connections

Let $X := \mathbb{P}^1$, with a divisor consisting of four distinct points $D := \{t_1, t_2, t_3, t_4\}$, and put $U := X - D$. Let $\mathcal{M}^d \to \mathbb{A}^1$ denote the moduli stack [23, 24] of logarithmic λ-connections of rank two and degree d with quasiparabolic structure on (X, D). For a scheme S, an object of $\mathcal{M}(S)$ is a quadruple $(\lambda, E, \nabla, P_\bullet)$ where $\lambda : S \to \mathbb{A}^1$, E is a rank 2 vector bundle on $X \times S$ of degree d on the fibers $X \times \{s\}$, $P_\bullet = (P_1, P_2, P_3, P_4)$ is a collection of rank 1 subbundles

$$P_i \subset E|_{\{t_i\}\times S},$$