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TOWARD ABHYANKAR'S INERTIA CONJECTURE FOR $PSL_2(\ell)$

by

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Abstract. — For $\ell \neq p$ odd primes, we examine $PSL_2(\ell)$-covers of the projective line branched at one point over an algebraically closed field k of characteristic p, where $PSL_2(\ell)$ has order divisible by p. We show that such covers can be realized with a large variety of inertia groups. Furthermore, for each inertia group realized, we can realize all "sufficiently large" higher ramification filtrations.

Résumé **(Vers la conjecture d'inertie d'Abhyankar pour $PSL_2(\ell)$).** — Pour ℓ premier impair distinct de p, nous étudions les $PSL_2(\ell)$-revêtements de la droite projective ramifiés au-dessus d'un seul point sur un corps algébriquement clos k de caractéristique p, où p divise l'ordre de $PSL_2(\ell)$. Nous montrons que de tels revêtements peuvent être réalisés avec une grande variété de groupes d'inertie. De plus, pour chaque groupe d'inertie réalisé, nous pouvons réaliser toutes les ramifications supérieures "suffisamment grandes".

1. Introduction

Over an algebraically closed field k of characteristic 0, finite algebraic branched covers $Y \to \mathbb{P}^1$ with n fixed branch points are in one-to-one correspondence with finite topological branched covers of the Riemann sphere with n fixed branch points. Both correspond to finite index subgroups of $\pi_1(\mathbb{P}^1_k \backslash \{x_1, \ldots, x_n\})$, the free (profinite) group on $n-1$ generators. In particular, there exist no nontrivial covers of $\mathbb{P}^1_k$ branched at one point (hereafter called "one-point covers").

If k is algebraically closed of characteristic p, the situation differs in two important ways. First, there exist many covers which do not have topological analogs. In

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particular, as a consequence of *Abhyankar's conjecture* (proven by Raynaud ([**Ray94**]) in the case of the affine line and Harbater ([**Har94**]) in general), it follows that a finite group G can be realized as the Galois group of an n-point cover of $\mathbb{P}^1_k$ exactly when $G/p(G)$ can be generated by $n-1$ elements (here, $p(G)$ is the subgroup of G generated by all of the p-Sylow subgroups). Thus, G occurs as the Galois group of a one-point cover iff $G = p(G)$. Such a group is called *quasi-p*. The second major difference is that, unlike in characteristic zero, the genus of Y is not determined by the degree, ramification points, and ramification indices of $f : Y \to \mathbb{P}^1$, but also depends on wild ramification behavior at the ramification points. This behavior is encoded in the higher ramification filtration (§2.1).

While Abhyankar's conjecture guarantees existence of certain one-point G-covers, it does not provide examples. In particular, the question of what subgroups $I \subseteq G$ can occur as the inertia group of a G-Galois one-point cover has been answered in only a few cases. By basic ramification theory, such a group I must be of the form $P \rtimes \mathbb{Z}/m$, where P is a p-group and $p \nmid m$. Furthermore, I must also generate G as a normal subgroup (this is automatic, for instance, when G is simple). *Abhyankar's inertia conjecture* ([**Abh01**]) states that any I satisfying these properties occurs as the inertia group of a one-point G-cover. This is known to be true for $G = PSL_2(p)$ and $G = A_p$, for $p \geq 5$ ([**BP03**]), and for A_{p+2} when $p \equiv 2 \pmod 3$ is an odd prime ([**MP10**]). It is clearly true in some trivial cases, for instance when G is abelian. However, outside of these cases, our knowledge is limited. Abhyankar has constructed many examples of one-point G-covers with various inertia groups where G is simple, but all of these examples are in cases where G is alternating, sporadic, or of Lie type over a field of characteristic p. In particular, if G is a simple group with a cyclic p-Sylow subgroup of order greater than p (such a group cannot be alternating, sporadic, or of Lie type over a field of characteristic p), then as far as I know, no examples of one-point G-covers have been constructed, let alone with any particular inertia group.

In this paper, we investigate Abhyankar's inertia conjecture for certain simple groups $G = PSL_2(\ell)$, where $p \,|\, |G|$ and $\ell \neq p$ are odd primes. In this case, the group $PSL_2(\ell)$ has a cyclic p-Sylow subgroup of order p^a, where $a = v_p(\ell^2 - 1)$. While we are not able to prove the conjecture in full (see Remark 3.8), we exhibit many examples with a wide variety of inertia groups:

***Corollary* 3.6.** — *Let k be an algebraically closed field of characteristic $p \geq 7$ and $G \cong PSL_2(\ell)$, where $p \,|\, |G|$. Suppose I is either a cyclic group of order p^r or a dihedral group D_{p^r} of order $2p^r$, with $1 \leq r \leq v_p(|G|)$. Then there exists a G-cover $f : Y \to \mathbb{P}^1_k$ branched at one point with inertia groups isomorphic to I.*

This result generalizes [**BP03**, Theorem 3.6], which deals with the case $a = 1$. We also investigate the higher ramification behavior of one-point $PSL_2(\ell)$-covers with inertia group I. We show that, in the situation of Corollary 3.6, any sufficiently "large" higher ramification filtration must occur. See Corollary 4.5 for a more specific result.

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2. Preliminaries

Notation: If G is a group with cyclic p-Sylow subgroup P, we write m_G for $|N_G(P)/Z_G(P)|$, the normalizer modulo the centralizer. If R is a local ring or a discretely valued field, then $\hat{R}$ is the usual completion. If x is a point of a scheme X, then $\mathcal{O}_{X,x}$ is the local ring of X at x.

2.1. Higher ramification filtrations. — We recall some facts from [**Ser79**, IV]. Let K be a complete discrete valuation field with algebraically closed residue field k of characteristic $p > 0$. If L/K is a finite Galois extension of fields with Galois group G, then L is also a complete discrete valuation field with residue field k. Here G is of the form $P \rtimes \mathbb{Z}/m$, where P is a p-group and m is prime to p. The group G has a filtration $G \supseteq G^i$ ($i \in \mathbb{R}_{\geq 0}$) called the *higher ramification filtration for the upper numbering*. If $i \leq j$, then $G^i \supseteq G^j$ (see [**Ser79**, IV, §1, §3]). The subgroup G^i is known as the *ith higher ramification group for the upper numbering*. One knows that $G^0 = G$, and that for sufficiently small $\epsilon > 0$, $G^\epsilon = P$. For sufficiently large i, $G^i = \{id\}$. Any i such that $G^i \supsetneq G^{i+\epsilon}$ for all $\epsilon > 0$ is called an *upper jump* of the extension L/K. If i is an upper jump and $i > 0$, then $G^i/G^{i+\epsilon}$ is an elementary abelian p-group for sufficiently small ϵ. The greatest upper jump (i.e., the greatest i such that $G^i \neq \{id\}$) is called the *conductor* of higher ramification of L/K.

If $P \cong \mathbb{Z}/p^r$ is cyclic, then G must have r different positive upper jumps $u_1 < \cdots < u_r$. Since the sequence $(u_1, \ldots, u_r)$ encodes the entire higher ramification filtration, we will simply say that such an extension (or the inertia group of such an extension) has upper higher ramification filtration $(u_1, \ldots, u_r)$ in this case.

The higher ramification filtration is important because if $f : Y \to X$ is a branched cover of curves in characteristic p, and $y \in Y$ is a ramification point and $f(y) = x$, then the higher ramification filtration for $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{X,x}$ figures into the ramification divisor term in the Hurwitz formula. Specifically, let $f : Y \to \mathbb{P}^1$ be a one-point G-cover with inertia groups $I \cong \mathbb{Z}/p^r \rtimes \mathbb{Z}/m$ with $p \nmid m$ and higher ramification filtration

$(u_1, \ldots, u_r)$. Then, by [Pr06, Lemma 1], the genus of Y is $1 - |G| + \frac{|G|\deg(R)}{2mp^r}$, where R is the ramification divisor, which has degree

$$mp^r - 1 + (p-1)m(u_1 + pu_2 + \cdots + p^{r-1}u_r).$$

2.2. Stable reduction. — Let $X \cong \mathbb{P}^1_K$, where K is a characteristic zero complete discretely valued field with algebraically closed residue field k of characteristic $p > 0$ (e.g., K is the completion of the maximal unramified extension of $\mathbb{Q}_p$). Let R be the valuation ring of K.

Let $f : Y \to X$ be a G-Galois cover defined over K, with G any finite group, such that the branch points of f are defined over K and their specializations do not collide on the special fiber of X_R. Assume that $2g_X - 2 + r \geq 1$, where g_X is the genus of X and r is the number of branch points of f. By a theorem of Deligne and Mumford ([DM69, Corollary 2.7]), combined with work of Raynaud ([Ray99]) and Liu ([Liu06]), there is a minimal finite extension K^{st}/K with ring of integers R^{st}, and a unique model Y^{st} of $Y_{K^{st}}$ (called the *stable model*) such that

- The special fiber $\overline{Y}$ of Y^{st} is semistable (i.e., it is reduced, and has only nodes for singularities).
- The ramification points of $f_{K^{st}} = f \times_K K^{st}$ specialize to *distinct* smooth points of $\overline{Y}$.
- Any genus zero irreducible component of $\overline{Y}$ contains at least three marked points (i.e., ramification points or points of intersection with the rest of $\overline{Y}$).

Since the stable model is unique, it is acted upon by G, and we set $X^{st} = Y^{st}/G$. Then X^{st} is semistable ([Ray90]) and can be naturally identified with a blowup of $X \times_R R^{st}$ centered at closed points. The map $f^{st} : Y^{st} \to X^{st}$ is called the *stable model* of f. The special fiber $\overline{f} : \overline{Y} \to \overline{X}$ of f^{st} is called the *stable reduction* of f.

From now on, assume that f has bad reduction (i.e., $\overline{X}$ is reducible). Any irreducible component $\overline{X}_b$ of $\overline{X}$ that intersects the rest of $\overline{X}$ in only one point is called a *tail*. If G acts without inertia above the generic point of $\overline{X}_b$, we call $\overline{X}_b$ an *étale tail*. If, furthermore, $\overline{X}_b$ contains the specialization of a branch point of f, then $\overline{X}_b$ is called *primitive*. If not, $\overline{X}_b$ is called *new*. On an étale tail $\overline{X}_b$, branching of $\overline{f}$ can only take place at the point where $\overline{X}_b$ intersects the rest of $\overline{X}$ and, if $\overline{X}_b$ is primitive, where the branch point of f specializes.

Suppose G has a cyclic p-Sylow subgroup P. Consider an étale tail $\overline{X}_b$ of $\overline{X}$. Let $\overline{x}_b$ be the unique point at which $\overline{X}_b$ intersects the rest of $\overline{X}$. Let $\overline{Y}_b$ be a component of $\overline{Y}$ lying above $\overline{X}_b$, and let $\overline{y}_b$ be a point lying above $\overline{x}_b$. Then the *effective ramification invariant* σ_b is the conductor of higher ramification of the extension $\mathrm{Frac}(\hat{\mathcal{O}}_{\overline{Y}_b,\overline{y}_b})/\mathrm{Frac}(\hat{\mathcal{O}}_{\overline{X}_b,\overline{x}_b})$.