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by

Marius van der Put

Abstract. — After explaining the problem and the results in a short survey of joint work with M-H. Saito and work by K. Okamoto on the geometry of Painlevé equations, two special families of linear differential equations, related to the equations PI, PIII(D_8), are studied in detail. Fine moduli spaces are constructed and identified with Okamoto–Painlevé spaces. The universal property of the moduli spaces implies the Painlevé property for these equations.

***Résumé* (Familles d'équations différentielles linéaires et les équations de Painlevé)**

Nous présentons rapidement le problème étudié et les résultats obtenus sur la géométrie des équations de Painlevé, par l'auteur et M–H. Saito d'une part, et par K. Okamoto d'autre part. Ensuite, deux familles spéciales en relation avec PI et PIII(D_8) sont étudiées en détail. On construit des espaces de modules fins et on les identifie aux espaces d'Okamoto–Painlevé. La propriété universelle de ces espaces de modules entraîne la propriété de Painlevé pour les équations PI et PIII(D_8).

Introduction

It is well known that isomonodromic families of linear differential equations induce solutions of Painlevé equations. A systematic study of these families and the corresponding monodromy spaces is the theme of [vdP-Sa]. Let a finite subset A of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and a map $r : A \to \mathbb{Q}_{\geq 0}$ be given. Consider the set $\mathbf{S}$ (of the isomorphy classes) of the differential modules M over $\mathbb{C}(z)$ satisfying:
(i) $\dim M = 2$ and $\Lambda^2 M$ is the trivial module,
(ii) A is the set of singular points of M and $r(a)$ is the Katz invariant of M at the point a.

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Associated to the data (A, r) there is a moduli space $\mathcal{R}$, called the *monodromy space*, which describes the analytic data, i.e., the ordinary monodromy, the Stokes maps and the links. There is also a space $\mathcal{P}$ of parameters, e.g., eigenvalues of the formal monodromies. The map $\mathcal{R} \to \mathcal{P}$ is a morphism between affine complex varieties.

The Riemann–Hilbert map $RH : \mathbf{S} \to \mathcal{R}$ associates to a given module $M \in \mathbf{S}$ its analytic data, i.e., a point of $\mathcal{R}$. A priori, $\mathbf{S}$ has no structure of algebraic variety. However, a preliminary structure of algebraic variety can be given to at least a subset of $\mathbf{S}$, by representing a suitable $M \in \mathbf{S}$ by a matrix differential operator of the form $\frac{d}{dz} + \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ for certain rational functions a, b, c (see [vdP-Sa]).

One requires that the fibers of $RH : \mathbf{S} \to \mathcal{R}$ have dimension 1. This leads to the ten cases for (A, r) of the following table.

Dynkin	Painlevé eqn	$r(0)$	$r(1)$	$r(\infty)$	$r(t)$	$\dim \mathcal{P}$
$\tilde{D}_4$	PVI	0	0	0	0	4
$\tilde{D}_5$	PV	0	0	1	-	3
$\tilde{D}_6$	$\mathrm{PV}_{\mathrm{deg}}(\mathrm{D}_6)$	0	0	1/2	-	2
$\tilde{D}_6$	$\mathrm{PIII}(\mathrm{D}_6)$	1	-	1	-	2
$\tilde{D}_7$	$\mathrm{PIII}(\mathrm{D}_7)$	1	-	1/2	-	1
$\tilde{D}_8$	$\mathrm{PIII}(\mathrm{D}_8)$	1/2	-	1/2	-	0
$\tilde{E}_6$	PIV	0	-	2	-	2
$\tilde{E}_7$	PII	0	-	3/2	-	1
$\tilde{E}_7$	PII	-	-	3	-	1
$\tilde{E}_8$	PI	-	-	5/2	-	0

The first row is the classical example by Schlesinger and Fuchs. Six of the remaining nine rows are known by Garnier, Fuchs, Jimbo, Miwa, Ueno, Flaschka, Newell et al. ([JMU, JM]). The three new rows were more recently found by Ohyama and Okumura, [OO].

The second column gives the associated Painlevé equation. The first column indicates the structure of Okamoto's space of initial conditions. Explicit calculations [vdP-Sa] show that $\mathcal{R} \to \mathcal{P}$ is a family of affine cubic surfaces.

The aim of this paper is to construct a fine moduli space $\mathcal{M}$ over $\mathbb{C}$ such that $\mathbf{S}$ *coincides in a natural way with $\mathcal{M}(\mathbb{C})$. Here we develop this theme for the families $(-,-,5/2)$ and $(1/2,-,1/2)$. The universal property of the fine moduli space implies the Painlevé property. It turns out that these fine moduli spaces $\mathcal{M}$ are the Okamoto–Painlevé space of the corresponding equations* $\mathrm{PI}, \mathrm{PIII}(\mathrm{D}_8)$.

In a sequel to this paper ([vdP2]) we construct fine moduli spaces for the two families $(0,-,3/2)$ and $(-,-,3)$. As one may expect, these spaces are again Okamoto–Painlevé spaces and the Painlevé property is proven for all PII equations. In another sequel to this paper, the families $(1,-,1/2)$ and $(1,-,1)$, related to

PIII(D_7) and PIII(D_6), are studied.

Proving the Painlevé property for the Painlevé equations has been the subject of much research. Major results are by M. Hukuhara [**OT**], B. Malgrange [**Mal1**, **Mal2**], M. Jimbo, T. Miwa and K. Ueno [**JMU**, **JM**, **Miw**]. The problem is recently solved according to [**C**], [**GLS**].

For the convenience of the reader we summarize (with a slight change of terminology) some facts on Okamoto–Painlevé spaces that we will need and refer to [**O1**, **O2**, **O3**, **OKSO**] and [**In**, **IIS1**, **IIS2**, **IISA**, **STT**, **S-Ta**, **STe**] for more details.

Fix a Painlevé equation P_* and fix its parameters. An *Okamoto–Painlevé space for* P_* is a tuple $(E, \pi, B, \mathcal{F})$ where $\pi : E \to B$ is an analytic fibration, the base B is 1-dimensional, non singular and simply connected. $\mathcal{F}$ is a foliation on E with 1-dimensional leaves. One poses the following conditions:
The leaves are the solutions of P_* (more precisely the solutions of the Hamiltonian system of P_*). Every leaf $\mathcal{F}_\alpha$ is transversal to the fibers of π and the restriction $\mathcal{F}_\alpha \to B$ of π to every leaf $\mathcal{F}_\alpha$ is an isomorphism.
(In the later paper [**OKSO**] the base space B is connected but not necessarily simply connected. The last condition is replaced by: the restriction $\mathcal{F}_\alpha \to B$ of π to every leaf $\mathcal{F}_\alpha$ is an unramified covering).

The fibers of $\pi : E \to B$ are analytically isomorphic and the *space of the initial conditions* denotes a fiber.

Okamoto constructs an Okamoto–Painlevé space for each P_* under the assumption that P_* has the Painlevé property. This property can be stated here as: Every local meromorphic solution of P_* extends to a meromorphic solution on the universal covering B of the space T of the values of t, where the equation P_* has no singularity.

The terminology 'space of initial conditions' can be explained as follows. Suppose for convenience that $B = T$. For a fixed $t_0 \in T$, the fiber $\pi^{-1}(t_0)$ meets every leaf $\mathcal{F}_\alpha$ with multiplicity one. Thus $\pi^{-1}(t_0)$ coincides with the collection of the local meromorphic solutions for P_* at t_0.

We make this explicit for P I $q'' = 6q^2 + 2t$ and P II $q'' = 2q^3 + tq + \alpha$.
At a point t_0 we consider all meromorphic solutions. The holomorphic solutions are uniquely determined by the initial values $q(t_0), q'(t_0)$. But there are also solutions with poles.

For P I $(t-t_0)^{-2} - \frac{t_0}{5}(t-t_0)^2 - \frac{1}{3}(t-t_0)^3 + h(t-t_0)^4 + \sum_{n>4} c_n(t-t_0)^n$ with h arbitrary and c_n polynomials in h.

For P II there are two types
$(t-t_0)^{-1} - \frac{t_0}{6}(t-t_0) - \frac{\alpha+1}{4}(t-t_0)^2 + h(t-t_0)^3 + \sum_{n>3} c_n(t-t_0)^n$ with h arbitrary and c_n polynomials in h.

$-(t-t_0)^{-1} + \frac{t_0}{6}(t-t_0) - \frac{\alpha-1}{4}(t-t_0)^2 + h(t-t_0)^3 + \sum_{n>3} c_n(t-t_0)^n$ with h arbitrary and c_n polynomials in h.

In the above examples $\pi^{-1}(t_0)$ is the union of $\mathbb{C}^2$ and one or two lines.

The space of initial conditions has, by Okamoto's construction, the form $S \setminus Y$ (depending on the choice of $b \in B$), where S is a complete surface and Y is a divisor. One calls (S, Y) an Okamoto–Painlevé pairs of non fiber type. The surface S is obtained by blowing up nine points (allowing for infinitely near points) of $\mathbb{P}^2$ which lie on a single curve $C \subset \mathbb{P}^2$ of degree three (or, equivalently, one blows up eight points on the Hirzebruch surface Σ_2). Further $Y \subset S$ is the unique anti-canonical divisor of S. The configuration of the irreducible components of Y for the Okamoto–Painlevé pairs of non fiber type does not depend on the choice of $b \in B$ and are given by the eight extended Dynkin diagrams $\tilde{D}_4, \tilde{D}_5, \tilde{D}_6, \tilde{D}_7, \tilde{D}_8,\ \tilde{E}_6,\ \tilde{E}_7,\ \tilde{E}_8$.

The method that we follow can be called 'The Riemann–Hilbert Approach'. Some of our results have parallels in [F], especially Chapters 5,6 and also in [OKSO].

1. The family $(-, -, 5/2)$

S denotes the set of the isomorphy classes of the differential modules M over $\mathbb{C}(z)$ satisfying: (i) $\dim M = 2$ and $\Lambda^2 M$ is the trivial module,
(ii) M has only at ∞ a singular point and $r(\infty) = 5/2$.
After changing the global variable z one may suppose that the formal module $\mathbb{C}((z^{-1/2})) \otimes M$ is represented by the matrix differential operator $z\frac{d}{dz} + \left(\begin{smallmatrix} w & 0 \\ 0 & -w \end{smallmatrix}\right)$ with $w = z^{5/2} + \frac{t}{2} z^{1/2}$.

The aim is to construct a connection $\nabla : \mathcal{V} \to \Omega(4[\infty]) \otimes \mathcal{V}$ which has a given $M \in \mathbf{S}$ as generic fiber. Thus $\mathcal{V}$ is a vector bundle of rank two with M as space of generic sections and $\nabla_{\frac{d}{dz}}$ coincides on M with the given derivation ∂ on M. The completion of the stalk of $\mathcal{V}$ at ∞, namely $\widehat{\mathcal{V}}_\infty$, is a $\mathbb{C}[[z^{-1}]]$-lattice in $\mathbb{C}((z^{-1})) \otimes M$, invariant under $z^{-2}\partial$ (with $\partial = \nabla_{\frac{d}{dz}}$) or invariant under $z^{-3}\delta$ (with $\delta = \nabla_{z\frac{d}{dz}}$). A $\mathbb{C}[[z^{-1}]]$-lattice $\Lambda \subset N := \mathbb{C}((z^{-1})) \otimes M$ is called *invariant* if $z^{-3}\delta(\Lambda) \subset \Lambda$.

On the other hand, any invariant lattice $\Lambda \subset N$ determines a unique connection $(\mathcal{V}, \nabla)$ with $\nabla : \mathcal{V} \to \Omega(4[\infty]) \otimes \mathcal{V}$, generic fiber M and $\widehat{\mathcal{V}}_\infty = \Lambda$ (see [vdP-Si], p. 176). Thus we need a classification of the invariant lattices of $N := \mathbb{C}((z^{-1})) \otimes M$.

1.1. A local computation. — Let N be a differential module over $\mathbb{C}((z^{-1}))$ such that $\mathbb{C}((z^{-1/2})) \otimes N$ has a representation by the matrix differential operator $\delta_N := \delta + \left(\begin{smallmatrix} w & 0 \\ 0 & -w \end{smallmatrix}\right)$, with $w = z^{5/2} + \frac{t}{2} z^{1/2}$ and $\delta = z\frac{d}{dz}$. As before, a lattice $\Lambda \subset N$ over $\mathbb{C}[[z^{-1}]]$ is called *invariant* if Λ is invariant under $z^{-3}\delta_N$.

For an invariant lattice Λ, the lattice $\Lambda^2(\Lambda) \subset \Lambda^2 N \cong (\mathbb{C}((z^{-1})), z\frac{d}{dz})$ has a basis b_Λ such that $\delta b_\Lambda = m b_\Lambda$ with $m \in \mathbb{Z}$. The integer $m(\Lambda) := m$ depends only on Λ and b_Λ is unique up to multiplication by an element in $\mathbb{C}^*$.