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by

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Abstract. — The aim of this note is to give a short overview of J. Kovacic's scheme-theoretic approach ([13], [14]) to the Galois theory of strongly normal differential extensions and to highlight the main difficulties and differences one has to overcome to obtain a similar theory for difference equations.

Résumé **(Sur la théorie de Galois des extensions différentielles et aux différences fortement normales)**

Dans cette note, nous donnons un bref panorama de l'approche schématique qu'a élaborée J. Kovacic de la théorie de Galois des extensions différentielles fortement normales, et nous mettons en évidence les principales difficultés et différences auxquelles conduit la recherche d'une théorie similaire pour les équations aux différences.

Introduction

As demonstrated by J. Kovacic in [13] and [14] the language of differential schemes provides an amazingly well-suited foundation for the differential Galois theory of strongly normal extensions. Based on his earlier work on differential schemes ([11], [12]) Kovacic was able to leave behind the universal differential field that was omnipresent in Kolchin's approach. But this is just a pleasant side-effect. In fact, there had already been approaches that go without a universal field ([1], [26]). In the opinion of the author, the main contributions of Kovacic have been

– a clear and direct way of constructing the differential Galois group as an algebraic group and
– his geometric characterization of strongly normal differential extensions ([14]).

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We will elaborate on the above two points. First of all we stress the fact that the notion of algebraic group used by Kovacic is the usual one, i.e., that of a group scheme of finite type over a field and not Kolchin's axiomatic one ([8]). The basic idea of Kovacic's approach is the following:

If $L|K$ is a strongly normal extension of differential fields with field of constants $k = K^\delta = L^\delta$ then the differential scheme δ-Spec$(L \otimes_K L)$ is split. By definition, this means that there exists a scheme $\mathcal{G}$ over k such that δ-Spec$(L \otimes_K L)$ is isomorphic (as δ-scheme over L) to the base extension of $\mathcal{G}$ from k to L. Moreover, this scheme $\mathcal{G}$ is the Galois group scheme of $L|K$ and if we formally set $Z = \delta$-Spec(L) then this isomorphism yields the familiar torsor-theorem

$$Z \times_K Z \simeq Z \times_k \mathcal{G}. \tag{1}$$

(Of course one first needs to work out the basic facts about products in the category of differential schemes.) The geometric characterization indicated above now states that the converse is also true. Namely, if $L|K$ is a finitely generated extension of differential fields with $L^\delta = K^\delta$ such that δ-Spec$(L \otimes_K L)$ is split then $L|K$ is strongly normal. Intuitively one might interpret this result by saying that the strongly normal extensions are precisely those extensions that admit a good Galois theory with *algebraic groups* as Galois groups. Of course there are Galois theories (e.g. [21], [25]) that apply to more general kind of extensions (or equations) but there the Galois groups are also more complicated objects then algebraic groups.

We emphasize the analogy with the classical Galois theory: A finite algebraic extension $L|K$ is Galois if and only if it splits over itself, i.e. Spec$(L \otimes_K L)$ is a set of L-rational points.

In Kovacic's approach the main challenge (the "heavy lifting" [14, p. 4139, Section 4]) is to show that δ-Spec$(L \otimes_K L)$ is split if $L|K$ is strongly normal. For this one has to come up with the Galois group scheme $\mathcal{G}$ somehow. While other approaches ([10], [1], [26]) relied on Weil's theory of "group chunks" to put the structure of an algebraic group on the δ-automorphisms of $L|K$ Kovacic directly defines $\mathcal{G}$ as the constants of $X = \delta$-Spec$(L \otimes_K L)$. Thus the underlying topological space of $\mathcal{G}$ is the same as the topological space of X and the structure sheaf of $\mathcal{G}$ is defined by $\mathcal{O}_\mathcal{G}(U) = \mathcal{O}_X(U)^\delta$ for every open subset U of X. With this definition it is easy to see that $\mathcal{G}$ is a locally ringed space but it remains the challenge to prove that $\mathcal{G}$ is a scheme.

We now describe the content of the article: In Section 1 we explain some of the most basic differences between differential and difference equations. In Section 2 we first recall the definition of strongly normal differential extensions. Our definition of a strongly normal difference extension is general enough to allow zero divisors and inseparability phenomena. So the Picard-Vessiot theory of [27] where the Picard-Vessiot extensions may well have zero divisors is included in our setup but our Galois group schemes need not be reduced. Our basic reference for the Galois theory of strongly normal difference extensions is [30]. One of the technical key results needed for establishing the fundamental isomorphism (1) – in the differential, as well as in the difference case – is that a strongly normal extension is constrained. In [30] a somewhat lengthy proof of this fact had to be given because certain difference analogs of results from differential algebra had been missing. Here we will present a shorter proof based on a difference algebraic Chevalley Theorem that has recently become available.

In Section 3 we recall the basic definitions from the theory of differential schemes and we explain the appropriate definitions for the difference case that will facilitate the fundamental isomorphism (1) also in the difference world. We also present a difference analog of Kovacic's geometric characterization of strongly normal extensions enriched by a third equivalent condition related to representable functors. Furthermore we show how the basic identity $\mathrm{trdeg}(L|K) = \dim(\mathcal{G})$ can be derived from the fundamental isomorphism in the difference setting.

If K is a difference field such that the endomorphism $\sigma : K \to K$ defining the difference structure is not surjective then one can embed K into the inversive closure K^* of K on which σ is an automorphism. In Section 4 we relate the Galois group of a system of linear difference equations over K to the Galois group of the system over K^*.

I would like to dedicate this article to the memory of Jerald J. Kovacic. Unfortunately I did not get the opportunity to meet him and to tell him how inspiring his articles have been for my work.

1. σ versus δ

The following conventions will be in force throughout the text: All rings and algebras are commutative. A differential ring (or δ-ring for short) is a ring, containing the field of rational numbers, equipped with a derivation δ. In particular all δ-fields are of characteristic zero.[1] However, our difference fields and rings are allowed to be of arbitrary characteristic. A difference ring (or σ-ring) is a ring together with a ring endomorphism σ (not necessarily injective or an automorphism). Basic references for difference algebra are [4] and [15].

One of the rather basic differences between differential and difference algebra is the following: If $L|K$ is a separable algebraic field extension then every derivation on K has a unique extension to a derivation on L (see e.g. [2, Chapter V, § 16, Prop. 4, p. A.V.129]). Thus in the study of differential field extensions such that the underlying extension of fields is separable algebraic we can in principle discard the derivation and deal with the field extension alone.

In the difference case the situation is much more challenging: Not every endomorphism of K extends to an endomorphism of L and if such an extension exists it

[1] In positive characteristic one would need to use iterative derivations, see e.g. [20].

is usually not unique. The theory of difference fields such that the underlying field extension is algebraic is quite intricate (see e.g. [4, Chapter 7, Section 15]). Several of the difficulties in difference algebra that do not have a differential analog can be traced back to this fact, for example the problem of incompatible σ-field extensions ([4, Chapter 7, Theorem 8, p. 223]) or the problem with "extending specializations" ([4, Chapter 7, Theorem 4, p. 211 and the example following the Theorem]). Often one can avoid complications by restricting the attention to difference field extensions $L|K$ such that K is relatively algebraically closed or relatively separably algebraically closed in L. For example, this assumption has been quite common in the Galois theory of difference equations ([5], [7], [1]). But if one restricts to difference field extensions $L|K$ such that K is relatively algebraically closed in L then the Galois theory remains somewhat incomplete: Only the connected subgroups appear in the Galois correspondence and it is in general not possible to associate a minimal splitting *field* – the Picard-Vessiot extension – to a linear difference equation. In [27] these deficits have been resolved by employing a slight generalization of difference fields which we will call difference pseudo fields (see Definition 1.1 below).

A typical example of an intricacy in difference algebra that has no counterpart in differential algebra is the phenomenon of incompatibility of difference fields. Recall that two difference field extensions $L|K$ and $L'|K$ are called compatible if there exists a difference field extension of K that contains K-σ-isomorphic copies of L and L'. It is not difficult to show that if K is relatively separably algebraically closed in L then $L|K$ is compatible with any other extension $L'|K$ of σ-fields.

It is easy to see that any two extensions of δ-fields $L|K$ and $L'|K$ are compatible (in the obvious sense): Choose a δ-maximal δ-ideal $\mathfrak{m}$ in the δ-ring $L \otimes_K L'$ (i.e. a δ-ideal maximal in the set of all proper δ-ideals). Because such an ideal is prime the total quotient ring of $(L \otimes_K L)/\mathfrak{m}$ is a δ-field containing K-δ-isomorphic copies of $L|K$ and $L'|K$. The basic fact that a δ-maximal ideal is prime also implies that a δ-Picard-Vessiot ring is an integral domain and so its quotient field, the associated Picard-Vessiot extension is indeed a field.

In difference algebra a σ-maximal σ-ideal need not be prime, for example the ring of all periodic sequences in $\mathbb{C}$ with the shift operator is a σ-simple σ-ring that is not an integral domain. However, the structure of σ-simple σ-rings having only finitely many minimal prime ideals is manageable: If R is a σ-simple σ-ring with finitely many minimal prime ideals then R decomposes as a finite direct product of integral domains

$$R = R_1 \oplus \cdots \oplus R_d$$

and σ restricts to an injective morphism of rings $\sigma : R_i \to R_{i+1}$ for $i = 1, \ldots, d$ ($R_{d+1} := R_1$). See [27, Corollary 1.16, p.12] or [30, Proposition 1.1.2, p. 2]. In particular σ^d defines an endomorphism of R_i, i.e. R_i is a σ^d-ring. The total quotient ring K of R then is a difference ring of the form

$$K = K_1 \oplus \cdots \oplus K_d$$