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MOTIVATED CYCLES UNDER SPECIALIZATION

by

Anna Cadoret

Abstract. — This paper is essentially a survey of André's theory of pure motivated motives with an emphasis on specialization theory in characteristic zero. We review first the classical construction of pure motives and then turn to pure motivated motives whose construction is modeled upon the one of pure homological motives, replacing homological cycles by motivated cycles. Basically, motivated cycles are obtained from homological cycles by adjoining formally the Lefschetz involution so that the so-called standard conjectures become true in the category of pure motivated motives; in particular, this category is a semisimple Tannakian category naturally equipped with fibre functors coming from Weil cohomologies. The last section is devoted to the ℓ-adic version of André's specialization theorem for motivated cycles, which asserts that, given a family of motivated motives M over a scheme S of finite type over a finitely generated field k of characteristic 0, the locus of all $s \in S(k)$ where the motivated motivic Galois group associated with $M_{\overline{s}}$ degenerates is thin in $S(k)$. When S is a curve, we improve André's statement by resorting to a uniform open image theorem for ℓ-adic cohomology proved by A. Tamagawa and the author. We conclude by some applications of this specialization theorem.

***Résumé* (Spécialisation des cycles motivés).** — Cet article est une introduction à la théorie des motifs motivés purs développée par André. Nous nous intéressons plus particulièrement au problème de la spécialisation de ces motifs en caractéristique 0. Nous commençons par rappeler la construction classique des motifs purs puis nous présentons la construction des motifs purs motivés comme une variante de la construction des motifs purs homologiques où les cycles homologiques sont remplacés par les cycles motivés. En gros, les cycles motivés sont obtenus en adjoignant formellement l'involution de Lefschetz aux cycles homologiques de sorte que les conjectures dites standard deviennent vraies dans la catégorie des motifs purs motivés; en particulier, cette catégorie est une catégorie tannakienne semisimple naturellement munie de foncteurs fibres provenant des cohomologies de Weil considérées. La dernière partie de cet article est consacrée à la version ℓ-adique du théorème d'André sur la spécialisation des cyces motivés. Celui-ci peut s'énoncer comme suit. Soit k un corps de type fini et de caractéristique nulle, S un schéma de type fini sur k et M une famille de motifs motivés sur S. Alors l'ensemble des points $s \in S(k)$ où le groupe de Galois motivé motivique associé à $M_{\overline{s}}$ dégénère est mince dans $S(k)$. Lorsque S est une courbe, nous améliorons le résultat d'André en invoquant un théorème d'image ouverte uniforme

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du à A. Tamagawa et l'auteur. Nous concluons en donnant quelques applications de ce théorème de spécialisation.

1. Introduction

Classically, (pure) motives can be presented either as an attempt to construct a universal cohomology or as an attempt to "embed" the category of smooth projective varieties into a neutral semisimple Tannakian category over a field E. Both points of view are intrinsically connected but we will rather adopt the second one, which is more adapted to André's theory of motivated motives.

Recall that a neutral Tannakian category over a field E is a rigid abelian tensor category which admits a faithful tensor functor with value in the category of finite dimensional E-modules. The main theorem of Tannakian formalism asserts that a neutral Tannakian category is equivalent to the category of finite dimensional E-rational representations of a pro-algebraic group over E (pro-reductive if the category is furthermore assumed to be semisimple).

Fix a field k of characteristic 0 and let $\mathcal{P}(k)$ denote the category of smooth, projective schemes over k and $\mathcal{P}(k)^{op}$ its opposite category. These are tensor categories. The first step of the construction of pure motives is to "embed" $\mathcal{P}(k)^{op}$ into an additive tensor category - this is the category of homological correspondences. Once one has an additive tensor category, one can, in turn, "embed" it into its Karoubian enveloppe, which is a pseudoabelian tensor category - this is the category of effective motives. The third step consists in inverting formally the so-called Lefschetz motive to obtain the category of pure motives, which is a rigid pseudoabelian tensor category. The category of pure motives, however, is not Tannakian yet and, unfortunately, the remaining part of the construction is only conjectural, based on the so-called standard conjectures. These conjectures are all implied by the so-called Lefschetz type conjecture, which predicts that the Lefschetz involution is a morphism in the category of pure motives.

The key idea of André's construction of motivated cycles is to adjoin formally the Lefschetz involutions to the set of homological correspondances in order to force Lefschetz conjecture to hold and construct a category of motives which is a semisimple neutral Tannakian category - the category of pure motivated motives. In particular, to any $X \in \mathcal{P}(k)$ one can associate the tensor subcategory $\langle X\rangle^{\otimes}$ generated by X in the category of pure motivated motives; this is again a semisimple neutral Tannakian category and its Galois group $G_{mot}(X)$ is a reductive algebraic group.

Now, given a scheme S, smooth, separated and geometrically connected over k with generic point η and a smooth projective morphism $f : X \to S$ with geometrically connected fibres, one can ask how the categories $\langle X_{\overline{s}}\rangle^{\otimes}$ vary with $s \in S$ or, equivalently, how the $G_{mot}(X_{\overline{s}})$ do. This problem is dealt with in [A96, §5].

First, one has to find a way to compare $\langle X_{\overline{s}}\rangle^{\otimes}$ and $\langle X_{\overline{\eta}}\rangle^{\otimes}$, $s \in S$. Using the semisimplicity of the category of pure motivated motives and Deligne's fixed part theorem, one can show that the specialization isomorphism for ℓ-adic cohomology

$$sp_s : \bigoplus_{i\geq 0} \mathrm{H}^{2i}(X_{\overline{\eta}}, \mathbb{Q}_\ell)(i) \tilde{\rightarrow} \bigoplus_{i\geq 0} \mathrm{H}^{2i}(X_{\overline{s}}, \mathbb{Q}_\ell)(i)$$

maps motivated cycles to motivated cycles (corollary 4.7). So, as motivated motivic Galois groups are reductive, one can identify $G_{mot}(X_{\overline{s}})$ with a subgroup of $G_{mot}(X_{\overline{\eta}})$ and equality holds if and only if the specialization morphism for ℓ-adic cohomology induces an isomorphism onto motivated cycles for all fibre power $X \times_k X \times_k \cdots \times_k X$.

The next natural question is to understand the structure of the set of all $s \in S$ such that $G_{mot}(X_{\overline{s}}) \subsetneq G_{mot}(X_{\overline{\eta}})$; André's specialization theorem for motivated cycles answers it, at least partially.

Theorem 1.1. — ([**A96**, Thm. 5.2]) *For any finite field extension k'/k the set of all $s \in S(k')$ such that*

$$G_{mot}(X_{\overline{s}}) \subsetneq G_{mot}(X_{\overline{\eta}})$$

is thin in $S(k')$.

The proof is along the following guidelines. First, one observes that $G_{mot}(X_{\overline{s}})$ contains an open subgroup of the image G_s of the ℓ-adic Galois representation

$$\rho_{f,s} : \Gamma_{k(s)} \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{s}}, \mathbb{Q}_\ell)).$$

As a result, the degeneration of $G_{mot}(X_{\overline{s}})$ forces the degeneration of G_s. Similarly, $G_{mot}(X_{\overline{\eta}})$ contains an open subgroup of the image G of the generic ℓ-adic Galois representation

$$\rho_{f,\eta} : \Gamma_{k(\eta)} \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Q}_\ell))$$

and identifying $\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ and $\mathrm{H}^*(X_{\overline{s}}, \mathbb{Q}_\ell)$ *via*

$$sp_s : \bigoplus_{i\geq 0} \mathrm{H}^{2i}(X_{\overline{\eta}}, \mathbb{Q}_\ell)(i) \tilde{\rightarrow} \bigoplus_{i\geq 0} \mathrm{H}^{2i}(X_{\overline{s}}, \mathbb{Q}_\ell)(i),$$

one can regard G_s as a closed subgroup of G. So, the set where $G_{mot}(X_{\overline{s}}) \subsetneq G_{mot}(X_{\overline{\eta}})$ is contained in the set where G_s is not open in G. The problem thus amounts to studying this second set.

To control this second set, André resorts to a profinite variant of Serre's irreducibility theorem [**Se89**, p.148]. When S is a curve, this can be replaced by a uniform open image theorem for ℓ-adic representions of étale fundamental groups proved by A. Tamagawa and the author ([**CT09b**, Thm. 1.1] - see theorem 5.3) to obtain the following. Given an integer $d \geq 1$, let $S^{\leq d}$ denote the set of all closed points $s \in S$ such that $[k(s) : k] \leq d$.

Theorem 1.2. — *Assume that S is a curve and that k is a finitely generated field of characteristic 0. Then, for any integer $d \geq 1$, the set of all $s \in S^{\leq d}$ such that $G_{mot}(X_{\overline{s}}) \subsetneq G_{mot}(X_{\overline{\eta}})$ is finite.*

The paper is organized as follows. In section 2, we review the construction of the category of pure motives (after some preliminaries - gathered in subsection 2.1 - about algebraic cycles and Weil cohomologies) and, in section 3, we discuss the formalism of the standard conjectures. In section 4, we give the main features of André's theory of motivated cycles and explain how to specialize them. Section 5 is devoted to the statement and proof of the specialization theorem for motivated motivic Galois groups (theorem 5.1, which gathers theorem 1.1 and 1.2). We conclude this last section by discussing related topics such as jumping of the Neron-Severi rank or Tate conjectures.

It goes without saying that I am very much indebted to the reading of André's works [**A96**] and [**A04**] for the writing of this paper. I am also grateful to the referee for his or her constructive remarks.

2. The category of pure motives

2.1. Algebraic cycles and Weil cohomologies. — The aim of this preliminary section is to review the formalism of algebraic cycles and Weil cohomologies required to introduce the category of algebraic correspondences, which is the starting point of the construction of the category of pure motives. The content here is very standard and can be skipped by any reader familiar with these notions. We assume basic knowledge about intersection theory [**F84**], usual Weil cohomologies (say Betti and ℓ-adic) and Tannakian formalism [**DM82**].

Given a field K, we write $\mathrm{Mod}_{/K}$ for the category of K-modules, $\mathrm{Mod}_{/K}^{\mathbb{Z}_{\geq 0}}$ for the category of $\mathbb{Z}_{\geq 0}$-graded K-modules, $\mathrm{Alg}_{/K}^{\mathbb{Z}_{\geq 0}}$ for the category of $\mathbb{Z}_{\geq 0}$-graded K-algebras and $\mathrm{AAlg}_{/K}^{\mathbb{Z}_{\geq 0}}$ for the category of anticommutative $\mathbb{Z}_{\geq 0}$-graded K-algebras regarded as a $\otimes$-category whose commutativity constraint is given by Koszul rule that is, for any two $\mathbb{Z}_{\geq 0}$-graded algebras $M = \oplus_{i\geq 0} M_i$, $N = \oplus_{i\geq 0} N_i$, the commutativity constraint

$$c_{M,N} : M \otimes_K N \tilde{\to} N \otimes_K M$$

can be written as

$$c_{M,N} = \bigoplus_{i,j\geq 0} c_{i,j},$$

where

$$\begin{array}{rccccl} c_{i,j}: & M_i \otimes_K N_j & \tilde{\to} & N_j \otimes_K M_i & , & i,\ j \geq 0. \\ & m_i \otimes n_j & \mapsto & (-1)^{ij} n_j \otimes m_i & & \end{array}$$

Fix a field k, of characteristic 0.

Given a connected $X \in \mathcal{P}(k)$, we will write d_X for its dimension. Some statements involving d_X below only make sense if X is equidimensional. We will not necessarily