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NOTE ON TORSION CONJECTURE

by

Anna Cadoret & Akio Tamagawa

Abstract. — In this note, we give an elementary and effective proof of the fact that the torsion conjecture for jacobian varieties implies the torsion conjecture for abelian varieties.

***Résumé* (Note sur la conjecture de la torsion).** — Nous prouvons de manière élémentaire et effective que la conjecture de la torsion pour les variétés jacobiennes implique la conjecture de la torsion pour les variétés abéliennes.

1. Introduction

The classical torsion conjecture for abelian varieties over number fields can be stated as follows.

Conjecture 1.1. — *Let $d \geq 0$ be an integer then:*

- *(Weak form): Given a number field k, there exists an integer $N := N(k,d) \geq 0$ such that for any d-dimensional abelian variety A over k, one has:*

$$A(k)_{tors} \subset A[N].$$

- *(Strong form): Given an integer $\delta \geq 1$, there exists an integer $N := N(\delta,d) \geq 0$ such that for any number field k of degree $\leq \delta$ and any d-dimensional abelian variety A over k, one has:*

$$A(k)_{tors} \subset A[N].$$

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Completing a body of works initiated by B. Mazur in the mid-1970's [Ma77], L. Merel achieved a proof of the $d = 1$ case of the strong torsion conjecture in the mid-1990's [Me96]. But the $d > 1$ case remains widely open though recent results of the authors show that the strong torsion conjecture for the p-primary part of the torsion holds for d-dimensional abelian varieties parametrized by curves [CT09].

The aim of this note is to give a proof of the following statement, which, in particular, shows that the torsion conjecture for abelian varieties is equivalent to the torsion conjecture for jacobian varieties.

Theorem 1.2. — *Let $d > 0$ be an integer. Then there exists an integer $g(d) > 0$ satisfying the following property: For any infinite field k and any d-dimensional principally polarized abelian variety A over k there exists a smooth, geometrically connected curve $C \hookrightarrow A$ of genus $g_C \leq g(d)$ that induces a smooth surjective homomorphism with connected kernel $J_{C|k} \to A$ and a closed immersion $A \to J_{C|k}$ of abelian varieties.*

More precisely, one may take $C \hookrightarrow A$ of genus $g_C = g(d)$ with:

$$g(d) = 1 + 6^d (d-1)! \frac{d(d-1)}{2}.$$

Note that if A is an arbitrary (i.e., a not necessarily principally polarized) d-dimensional abelian variety over k then, by Zarhin's trick, $(A \times A^\vee)^4$ is an $8d$-dimensional principally polarized abelian variety over k. In particular, to prove the torsion conjecture for d-dimensional abelian varieties, it is enough to prove it for $g(8d)$-dimensional jacobian varieties.

Roughly speaking, the curve C in the statement of theorem 1.2 is constructed by cutting $d-1$ times A by "nice" hyperplanes. For the proof of the crucial fact that C can be chosen to have genus bounded only in terms of d, we give two different approaches in section 2 and section 3 respectively. The first approach given in section 2 is very elementary and relies on a certain explicit genus computation. More precisely, in subsection 2.1, we show that given a smooth, geometrically connected projective variety X over an infinite field k and a fixed embedding $X \hookrightarrow \mathbb{P}^n_k$, the curves obtained by cutting X by "nice" hyperplanes all have the same Hilbert polynomial, which depends only on the Hilbert polynomial of X. We then compute in subsection 2.2 effectively the Hilbert polynomial of the resulting curves. The second approach given in section 3 is more conceptual and relies on the theory of Hilbert schemes. More precisely, in subsection 3.1 we show that given a scheme S and a closed subscheme $X \hookrightarrow \mathbb{P}^n_S$ smooth, geometrically connected and (purely) of relative dimension $d > 0$ over S there exists a surjective smooth morphism $\pi : U \to S$ of finite type and a universal curve $q : \mathcal{C} \hookrightarrow X \times_S U \to U$ such that any curve constructed in X_s by cutting $d-1$ times X_s by "nice" hyperplanes arises as the fiber of q at some $h \in U_s$ (see proposition 3.1 for the precise statement). In subsection 3.2, we discuss the representability of the Hilbert functor. In subsection 3.3, we combine these results to recover the desired boundedness of the genus. Eventually, in section 4, we conclude the proof of theorem 1.2 by resorting to a weak version of Lefschetz theorem and a duality argument.

Remark 1.3. — 1. If k is a finite field, one may still show that for any (positive-dimensional) abelian variety A over k there exists a smooth, geometrically connected curve $C \hookrightarrow A$ for which the assertions of theorem 1.2 hold. Indeed, one has only to replace the classical hyperplane Bertini theorem by more recent hypersurface Bertini theorems due to O. Gabber [**G01**, Cor. 1.6] and B. Poonen [**P04**, Thm. 1.1] and note that lemma 4.1 also works for hypersurfaces. However, the genus of the curve constructed by this method is uncontrolled in Gabber's method and depends on the poles of the zeta functions of the successive sections in Poonen's method. So it is difficult to obtain a bound of the genus independent of the finite base field as in theorem 1.2.

2. If the characteristic of k is 0, our proof of theorem 1.2 is entirely elementary and classical. On the other hand, if the characteristic of k is positive, this argument only shows that the morphism $J_{C|k} \to A$ is surjective with connected kernel and the morphism $A \to J_{C|k}$ is finite with kernel having connected Cartier dual. To get the full statement in positive characteristic, we resort to [**G01**, Prop. 2.4], which may be less elementary.
3. The problem of how to realize abelian varieties as quotients of jacobian varieties is classical and the first proof of the fact that this can always be done (over an algebraically closed field) seems to go back to [**M52**]. Other references include [**Mi86**] and the already mentioned [**G01**].

We end this section by the following lemma, which is used in both of the two approaches.

Lemma 1.4. — *Let k be a field and A a d-dimensional abelian variety over k equipped with a polarization $\lambda : A \to A^\vee$ of degree δ^2 ($\delta > 0$). Let $\mathcal{L}$ denote the invertible sheaf $(id_A, \lambda)^*(\mathcal{P}_A)$ on A, where $\mathcal{P}_A$ is the normalized Poincaré sheaf on $A \times_k A^\vee$, so that $\phi_{\mathcal{L}} = 2\lambda$* [**MF82**, Chap.6, §2, Prop. 6.10], *and that $\mathcal{O}_A(1) := \mathcal{L}^{\otimes 3}$ is very ample relatively to $A \to k$* [**Mu70**, III, §17] *and induces a closed immersion $A \hookrightarrow \mathbb{P}^n_k$. Then the Hilbert polynomial $P(T)$ of A with respect to this embedding is given by:*

$$P(T) = 6^d \delta T^d.$$

In particular, $P(T)$ depends only on (d, δ).

Proof. This is stated in [**MF82**, Chap.7, §2] with a hint of proof. More explicitly, from the Riemann-Roch theorem [**Mu70**, III, §16], one has:

$$\chi(\mathcal{O}_A(n))^2 = \deg(\phi_{\mathcal{L}^{\otimes 3n}}) = \deg(3n\phi_{\mathcal{L}}) = \deg(6n\lambda) = (6n)^{2d}\deg(\lambda),$$

whence $\chi(\mathcal{O}_A(n)) = 6^d \delta n^d$ and $P(T) = 6^d \delta T^d$. $\square$

2. First approach — Genus computation

2.1. General case. — Let k be an infinite field and let $X_0 \hookrightarrow \mathbb{P}^n_k$ be a smooth, projective and geometrically connected variety of dimension $d > 0$ over k. If $d-1 > 0$, it follows from Bertini's theorem [J83, I, Th. 6.10 and Th. 7.1] that there exists a hyperplane $H_1 \hookrightarrow \mathbb{P}^n_k$ such that $X_1 := X_0 \times_{\mathbb{P}^n_k} H_1$ is a smooth, geometrically connected k-variety of dimension $d-1$. Iterating the process, one can construct a smooth geometrically connected k-variety X_i of dimension $d-i$ inductively for $0 \leq i \leq d-1$. More precisely, one set $X_i := X_{i-1} \times_{\mathbb{P}^n_k} H_i = X_0 \times_{\mathbb{P}^n_k} H_1 \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} H_i$. Also, let $\mathcal{O}_{X_i}(1)$ denote the very ample invertible sheaf relatively to k induced by the projective embedding $X_i \hookrightarrow \mathbb{P}^n_k$. Then, one has the short exact sequences of $\mathcal{O}_{X_i}$-modules:

$$0 \to \mathcal{O}_{X_i}(-1) \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_{i+1}} \to 0.$$

Hence, tensoring by $\mathcal{O}_{X_i}(n)$:

$$0 \to \mathcal{O}_{X_i}(n-1) \to \mathcal{O}_{X_i}(n) \to \mathcal{O}_{X_{i+1}}(n) \to 0,$$

from which it follows that $P_{i+1}(T) = P_i(T) - P_i(T-1)$, where $P_i(T)$ denotes the Hilbert polynomial of X_i. A straightforward inductive computation then yields:

$$P_i(T) = \sum_{0 \leq k \leq i} \binom{i}{k} (-1)^k P_0(T-k).$$

And, in particular, one can compute the genus $g(X_{d-1})$ of X_{d-1}:

$$g(X_{d-1}) = \dim_k(H^1(X_{d-1}, \mathcal{O}_{X_{d-1}})) = 1 - \sum_{0 \leq k \leq d-1} \binom{d-1}{k} (-1)^k P_0(-k).$$

Remark 2.1. — (Comparison with Castelnuovo's bound) By construction, the curves X_{d-1} obtained by cutting out X by $d-1$ hyperplanes all have the same degree as X - say a. From Castelnuovo's bound [ACGH85, p. 116], this implies that the genus of X_{d-1} is bounded from above by a constant

$$\pi(a,n) = \frac{q(q-1)}{2}(n-1) + qr,$$

where q and r denote the quotient and remainder of the euclidean division of $a-1$ by $n-1$ respectively.

2.2. The case of polarized abelian varieties. — We would like to apply the preceding computation to a d-dimensional abelian variety $X_0 = A$ over k equipped with a degree δ^2 polarization $\lambda : A \to A^\vee$ ($\delta > 0$). So, as in lemma 1.4, let $\mathcal{L}$ be the invertible sheaf on A such that $\phi_{\mathcal{L}} = 2\lambda$, hence $\mathcal{O}_A(1) := \mathcal{L}^{\otimes 3}$ is very ample relatively to $A \to k$ and induces a closed immersion $A \hookrightarrow \mathbb{P}^n_k$. Now, by lemma 1.4, the Hilbert polynomial $P_0(T)$ with respect to this embedding is given by: $P_0(T) = 6^d \delta T^d$. As a result:

$$g(X_{d-1}) = 1 + 6^d \delta (-1)^{d-1} \sum_{0 \leq k \leq d-1} \binom{d-1}{k} (-1)^k k^d.$$