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Emmanuel Lepage

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TEMPERED FUNDAMENTAL GROUP

by

Emmanuel Lepage

Abstract. — This paper is a survey of anabelian aspects of the tempered fundamental group of nonarchimedean analytic spaces. This tempered fundamental group classifies analytic étale coverings that become topological coverings for Berkovich topology after pullback by some finite étale covering. This article will focus on two aspects: a nonarchimedean analog of Grothendieck-Teichmüller theory and a geometric interpretation of compact subgroups of the tempered fundamental group and of a prime-to-p version of the tempered fundamental group.

***Résumé* (Groupe fondamental tempéré).** — Cet article est un survol des aspects anabéliens du groupe fondamental tempéré des espaces analytiques non-archimédiens. Ce groupe fondamental tempéré classifie les revêtements analytiques étales qui deviennent des revêtements topologiques après changement de base étale fini. Nous nous concentrerons sur deux aspects: un analogue non-archimédien de la théorie de Grothendieck-Teichmüller et une interprétation géométrique des sous-groupes compacts du groupe fondamental tempéré et d'une version première à p.

Introduction

For a complex algebraic variety, one can define a topological fundamental group by considering the associated analytic space; its profinite completion can be identified with the algebraic fundamental group. Y. André defined in [2] an analog of this topological fundamental group over nonarchimedean fields, called the tempered fundamental group. However, the topological fundamental group of a complex curve depends only of the genus of the curve, whereas the tempered fundamental group of a curve depends much more on the curve itself, even in the geometric case. For example, elliptic curves of good reduction and elliptic curves of bad reduction have

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non-isomorphic geometric tempered fundamental groups: in characteristic 0, the tempered fundamental group is $\widehat{\mathbf{Z}}^2$ in the case of good reduction, and $\mathbf{Z} \times \widehat{\mathbf{Z}}$ in the case of bad reduction. This paper sums up some anabelian aspects of the tempered fundamental group.

The tempered fundamental group is defined in terms of nonarchimedean analytic geometry. In this paper, the analytic setting will be expressed in terms of Berkovich spaces. Defining a fundamental group simply by considering the topology of a Berkovich space is not enough for our purposes: for example, the Berkovich space of $\mathbf{P}^1 \backslash \{0, 1, \infty\}$ is contractible, whereas its algebraic fundamental group is big enough—in the geometric case of characteristic 0, it is a free profinite group on two generators—to have very deep arithmetic properties. J. de Jong, in [11], introduced an étale fundamental group which classifies more general étale analytic coverings, and in particular some p-adic period maps in the sense of Rapoport and Zink. In the case of Lubin-Tate moduli spaces, studied by Gross and Hopkins in [9] and [8], one gets non trivial étale coverings of projective spaces, so that the fundamental group of projective spaces in the sense of de Jong is much more complicated than its counterpart in complex geometry. In this paper, we will focus on the tempered fundamental group of André. This group is much smaller and simpler than de Jong's fundamental group: for curves, the tempered fundamental group is residually finite, so that the map to its profinite completion—the algebraic fundamental group of Grothendieck—is injective. The tempered fundamental group is defined as the classifying group of the category of étale analytic coverings that become topological coverings after pullback by some finite étale covering. There is a universal pro-tempered covering: it is given by the projective system of universal topological coverings of the Berkovich space associated to a cofinal family of pointed finite étale coverings.

In this paper, we will be interested in anabelian aspects of the tempered fundamental group: what can be recovered of a variety from its tempered fundamental group? In arithmetic geometry, A. Grothendieck conjectured that a hyperbolic curve over a number field only depends up to isomorphism of its arithmetic fundamental group. More precisely, Grothendieck's conjecture predicts that, if X_1 and X_2 are two curves over a number field K, the map $\mathrm{Isom}(X_1, X_2) \to \mathrm{OutIsom}_{G_K}(\pi_1^{\mathrm{alg}}(X_{1,\overline{K}}, X_{2,\overline{K}}))$ is an isomorphism. This was proved by S. Mochizuki in [15]. In the tempered setting, one could hope for such anabelian properties even with geometric tempered fundamental groups, *i.e.* without restricting to equivariant isomorphisms of topological groups with respect to a Galois action. More precisely, a natural question could be the following: if X_1 and X_2 are two hyperbolic $\overline{\mathbf{Q}}_p$-curves, is the map

$\mathrm{Isom}_{\mathbf{Q}_p}(X_1, X_2) \to \mathrm{OutIsom}(\pi_1^{\mathrm{temp}}(X_{1,\mathbf{C}_p}), \pi_1^{\mathrm{temp}}(X_{2,\mathbf{C}_p}))$, given by functoriality of the tempered fundamental group, a bijection?

Grothendieck-Teichmüller theory tries to describe the absolute Galois group in terms of group of automorphisms of profinite fundamental groups of geometric objects. If the answer to the question of the last paragraph was positive, in the case $X_1 = X_2 = \mathbf{P}^1\backslash\{0, 1, \infty\}$, one would get a very precise non-Archimedean version of Grothendieck-Teichmüller theory: the absolute Galois group of $\mathbf{Q}_p$ would be equal to the group of outer automorphisms of $\mathbf{P}^1\backslash\{0, 1, \infty\}$ which sends each of the monodromy groups to a conjugate of itself. This is far from being known yet.

However, in [2], André started a non-Archimedean Grothendieck-Teichmüller theory. More precisely, he proved that every automorphism of the absolute Galois group of $\mathbf{Q}$ that induces an automorphism of $\pi_1^{\mathrm{alg}}(\mathbf{P}^1_{\overline{\mathbf{Q}}}\backslash\{0, 1, \infty\})$ which restricts to an automorphism of $\pi_1^{\mathrm{temp}}(\mathbf{P}^1_{\mathbf{Q}}\backslash\{0, 1, \infty\})$ is indeed in the absolute Galois group of $\mathbf{Q}_p$. By looking at the tower of moduli spaces $\mathcal{M}_{0,r}$ of curves of genus 0 with r ordered marked points, one defines a profinite group called the Grothendieck-Teichmüller group, which contains the absolute Galois group of $\mathbf{Q}$. André defines in [2] a p-adic analog of the Grothendieck-Teichmüller group, as a group of outer automorphisms of the geometric tempered fundamental group of $\mathbf{P}^1\backslash\{0, 1, \infty\}$.

We are very far from knowing how to compute the geometric tempered fundamental group of any hyperbolic curve over $\mathbf{C}_p$. However, a prime-to-p version of the tempered fundamental group, where p is the residue characteristic, is much easier to describe. In particular, for a hyperbolic curve over $\overline{\mathbf{Q}}_p$, S. Mochizuki gives in [17] a description of the compact subgroups in terms of the graph of the stable reduction. More precisely, there is a bijection between the set of conjugacy classes of maximal compact subgroups of the prime-to-p tempered fundamental group and the vertices of the graph of the stable reduction, and a bijection between the set of conjugacy classes of nontrivial intersection of two different maximal compact subgroups and the set of edges of the graph of the stable reduction. Thus one can recover the graph of the stable reduction from the prime-to-p tempered fundamental group. For the full tempered fundamental group, the maximal compact subgroups still appear as decomposition subgroups of points of the Berkovich space.

In the first part, we will define the topological space associated by Berkovich to an algebraic variety. We will recall that in the case of smooth curves, its homotopy type is encoded in the graph of a semistable reduction. Finally we will define the tempered fundamental group of an algebraic variety X as the projective limit of the Galois groups of the universal topological covering of Y over X, when Y runs over finite étale Galois coverings of X.

In a second part, we will be interested in Grothendieck-Teichmüller theory. In particular, we will sketch the proof that every automorphism of the absolute Galois group of $\mathbf{Q}$ that induces an automorphism of $\pi_1^{\mathrm{temp}}(\mathbf{P}^1_{\mathbf{Q}}\backslash\{0,1,\infty\})$ is in the absolute Galois group of $\mathbf{Q}_p$.

In the last part we will study the decomposition groups of Berkovich points of a curve over $\overline{\mathbf{Q}}_p$. First we will look at the images of these decomposition groups in the prime-to-p tempered fundamental group. In particular we will sketch the proof of the fact that one can recover the graph of the stable reduction of the curve from its prime-to-p tempered fundamental group. In the end, we will study maximal compact subgroups of the tempered fundamental group.

1. Tempered fundamental group

1.1. Berkovich analytification of algebraic varieties and curves. — Let K be a complete nonarchimedean field. We will mostly be interested later on in the case where $K = \mathbf{C}_p$ (the norm will be chosen so that $|p| = p^{-1}$ and the valuation so that $v(p) = 1$). In this paper, all valuations have values in $\mathbf{R} \cup \{\infty\}$.

If X is an algebraic variety over K, one can associate to X a topological set X^{an} with a continuous map $\phi : X^{\mathrm{an}} \to X$ defined in the following way.

A point of X^{an} is an equivalence class of morphisms $\operatorname{Spec} K' \to X$ over $\operatorname{Spec} K$ where K' is a complete valued extension of K. Two morphisms $\operatorname{Spec} K' \to X$ and $\operatorname{Spec} K'' \to X$ are equivalent if there exists a common valued extension L of K' and K'' such that

$$\begin{array}{ccc} \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} K'' \\ \downarrow & & \downarrow \\ \operatorname{Spec} K' & \longrightarrow & X \end{array}$$

commutes. In fact, for any point $x \in X^{\mathrm{an}}$, there is a unique smallest such complete valued field defining x: it is denoted by $\mathcal{H}(x)$ and is called the completed residue field of x. Forgetting the valuation, one gets points $\operatorname{Spec} K \to X$ from the same equivalence class of points: this defines a point of X, hence the map $X^{\mathrm{an}} \to X$. If $U = \operatorname{Spec} A$ is an affine open subset of X, every $x \in \phi^{-1}(U)$ defines a seminorm $|\ |_x$ on A. The topology on $\phi^{-1}(U)$ is defined to be the coarsest one such that $x \mapsto |f|_x$ is continuous for every $f \in A$.

For example, if K is algebraically closed, $a \in K$ and $r \in \mathbf{R}^{\geq 0}$, one gets a point $b_{a,r}$ of the analytic affine line defined by $|\sum_i a_i(T-a)^i|_{b_{a,r}} = \max_i |a_i| r^i$. However, unless K is spherically complete, not all points are of this type.