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by

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Abstract. — This article surveys aspects of stochastic process whose local nature varies with time. It first considers restrictions on the local scaling limits of processes if they are essentially unique. Then several methods are presented for constructing stochastic processes with prescribed local scaling limits which may vary with time. These include multifractional processes such as multifractional Brownian motion and multifractional stable processes. Then several constructions are presented for multistable processes, that is processes that are locally $\alpha(t)$-stable but where the stability index $\alpha(t)$ depends on t.

***Résumé* (Processus localisables, multifractionnaires et multistables).** — Cet article synthétise certains ascpects des processus stochastiques dont le comportement local varie avec le temps. Nous considérons tout d'abord les limites d'échelle locales, lorsque celles-ci sont uniques. Puis nous présentons plusieurs méthodes de construction de processus stochastiques dont les limites d'échelle locales varient en fonction du temps de manière particulière. Cet ensemble de processus comprend entre autres, des processus multifractionnaires tels que le mouvement brownien multifractionnaire et les processus stables multifractionnaires. Enfin nous présentons quelques constructions de processus multistables, c'est à dire des processus localement $\alpha(t)$-stables, dont l'indice d'autosimilarité $\alpha(t)$ dépend du temps.

1. Introduction

This article surveys recent work by J. Lévy Véhel, R. Le Guével, L. Liu and the author on the construction of stochastic processes $Y(t)\,(t \in \mathbb{R})$ that are localisable, in the sense that at each time t the process has a unique scaling limit or local form, but which might depend on t.

We first consider implications for the self-similarity and stationarity of increments of the local scaling limits when they are unique. We then discuss several methods for

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constructing stochastic processes with prescribed local scaling limits which may vary with the time. Such processes include multifractional processes such as multifractional Brownian motion and multifractional stable processes. We then discuss multistable processes, that is processes that are locally $\alpha(t)$-stable but where the stability index $\alpha(t)$ depends on t. Further possibilities include multifractional multistable processes, where both the scaling and stability indices vary. Such processes may be relevant, for example, in financial, traffic or terrain modelling.

Some of the results discussed here depend on rather lengthy technical conditions. We outline such results rather than give full statements and present illustrative examples. Readers requiring more detail are referred to papers where full proofs are given, particularly [**Fal6**, **FL**, **FLL**, **FLi**, **LL**].

2. Local structure of random processes

A number of authors, e.g. [**BJR**, **PL**, **ST1**], have considered the local form of random processes with slightly differing definitions and terminology. Given a process $Y(t)\,(t \in \mathbb{R})$ we call a random process W a *scaling limit at* $u \in \mathbb{R}$ of Y if there exist sequences $r_n \searrow 0$ and $c_n \searrow 0$ such that

$$\frac{Y(u+r_n t) - Y(u)}{c_n} \to W \tag{2.1}$$

as $n \to \infty$, where convergence is in finite dimensional distributions. If Y and W have versions in $C(\mathbb{R})$ (the space of continuous functions) or in $D(\mathbb{R})$ (the space of càdlàg functions) and convergence in (2.1) is in distribution (with respect to the metric of uniform convergence on bounded intervals, respectively the Skorohod metric), we call W a *strong scaling limit* of Y at u.

We call the family of possible scaling limits the *(strong) tangent space* of Y at u:

$$\mathrm{Tan}(Y,u) = \{W : W \text{ is a (strong) scaling limit of } Y \text{ at } u\}.$$

Clearly, if $W \in \mathrm{Tan}(Y,u)$ then $cW \in \mathrm{Tan}(Y,u)$ for all $c > 0$. In the nicest case there is some $W \neq 0$ such that

$$\mathrm{Tan}(Y,u) = \{cW\}_{c \geq 0},$$

and we say that Y is *(strongly) localisable* at u with *local form* W, and write $Y'_u = W$. In particular, if for some $h > 0$,

$$\frac{Y(u+rt) - Y(u)}{r^h} \to Y'_u(t)$$

as $r \searrow 0$ we call Y *(strongly) h-localisable* at u with *(strong) local form* Y'_u. Of course, strongly localisable processes are localisable.

Many standard processes are h-localisable. It is immediate from the definitions that sssi processes, that is processes which are *h-self-similar*, that is $Y(rt) = r^h Y(t)$ for

$r > 0$, and which have *stationary increments*, that is $Y(t+u) - Y(u) = Y(t) - Y(0) = Y(t)$ for $u \in \mathbb{R}$, are h-localisable.

Proposition 2.1. — *Let $Y(t)\ (t \in \mathbb{R})$ be a process that is h-self-similar with stationary increments. Then Y is h-localisable at all $u \in \mathbb{R}$ with $Y'_u = Y$. If in addition Y is in $C(\mathbb{R})$ or $D(\mathbb{R})$ then Y is strongly h-localisable at all $u \in \mathbb{R}$.*

Such sssi processes include index-h *fractional Brownian motion* (fBm) on $\mathbb{R}$ for $0 < h < 1$, which may be defined as a stochastic integral with respect to Wiener measure W:

$$B_h(t) = c(h)^{-1} \int_{-\infty}^{\infty} \left((t-x)_+^{h-1/2} - (-x)_+^{h-1/2}\right) W(dx),$$

where $(a)_+ = \max\{0, a\}$ and $c(h)$ is a normalising constant that ensures that the variance of $B_h(1)$ is 1. (Here, and throughout, we make the convention that expressions involving the difference of two positive parts represent the indicator function of the intermediate interval when the exponent is 0.) It is well-known [**EM**, **MV**, **ST**] that index-h fBm is an h-self-similar process with a version in $C(\mathbb{R})$ that has stationary increments, so is strongly localisable at all $u \in \mathbb{R}$ with $(B_h)'_u = B_h$.

Instead of integrals with respect to Wiener measure, we may consider α-stable integrals to get *α-stable processes*, that is processes Y for which the finite dimensional distributions $(Y(t_1), \ldots, Y(t_k))$ are α-stable vectors for all $t_1, \ldots, t_k$, see [**ST**] for definitions and properties of α-stable integrals. For example, linear stable fractional motion may be defined as

$$L_{\alpha,h}(t) = \int_{-\infty}^{\infty} \left((t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha}\right) M(dx), \tag{2.2}$$

where $0 < \alpha < 2$, $0 < h < 1$ and M is an α-stable random measure with constant skewness β and where, as throughout, we take M to have Lebesgue measure as control measure, see [**ST**]. The process is h-sssi and so is h-localisable at all $u \in \mathbb{R}$ with $(L_{\alpha,h})'_u = L_{\alpha,h}$. Provided that $h > 1/\alpha$ it has a version in $C(\mathbb{R})$, so is strongly localisable. However, if $h < 1/\alpha$ then almost surely Y is unbounded on every interval and so is not a process of $D(\mathbb{R})$, though it is nevertheless localisable.

An α-stable Lévy motion $(0 < \alpha < 2)$ is a process in $D(\mathbb{R})$ with stationary independent increments which have a strictly α-stable distribution. It may be represented as

$$L_\alpha(t) = M([0, t]) \tag{2.3}$$

where M is an α-stable random measure on $\mathbb{R}$ with constant skewness intensity, see [**ST**]. Then L_α is $1/\alpha$-sssi, and so is strongly $1/\alpha$-localisable.

The local form of an sssi process is the same process, and therefore the local form is sssi. By considering tangent spaces we can show that this is typical, that is that strong local forms are in general sssi. We summarise the invariance properties of tangent spaces.

Theorem 2.2. — *Let* $\mathrm{Tan}(Y,u)$ *be the strong tangent space of a process* Y *at* $u \in \mathbb{R}$. *Then*

1. *If* $W(t) \in \mathrm{Tan}(Y,u)$ *then*
$$cW(t) \in \mathrm{Tan}(Y,u) \text{ for all } c > 0 \qquad \text{(vertical scaling)},$$
2. *If* $W(t) \in \mathrm{Tan}(Y,u)$ *then*
$$W(rt) \in \mathrm{Tan}(Y,u) \text{ for all } r > 0 \qquad \text{(horizontal scaling)}\ ,$$
3. *For almost all* $u \in \mathbb{R}$, *if* $W(t) \in \mathrm{Tan}(Y,u)$ *then*
$$W(z+t) - W(z) \in \mathrm{Tan}(Y,u) \text{ for all } z \in \mathbb{R} \qquad \text{(translation invariance)}.$$

Proof. — Parts (1) and (2) are immediate from the definitions. Part (3) is more subtle, see [**Fal5**, **Fal6**]. □

This specialises easily to the localisable situation, where $\mathrm{Tan}(Y,u)$ contains essentially one process.

Corollary 2.3. — *Let* Y *be a random process in* $C(\mathbb{R})$ *or* $D(\mathbb{R})$. *For almost all* $u \in \mathbb{R}$ *at which* Y *is strongly localisable, the local form* Y'_u *is* h*-self-similar for some* $h > 0$ *and has stationary increments.*

The above corollary restricts the possible local forms of strongly localisable processes in a number of cases where the range of sssi processes are limited.

Corollary 2.4. — Gaussian processes. *Let* Y *be a Gaussian process in* $C(\mathbb{R})$. *For almost all* $u \in \mathbb{R}$ *at which* Y *is strongly localisable, the local form* Y'_u *is either index-*h *fractional Brownian motion for some* $0 < h < 1$, *or* $Y'_u(t) \equiv tW$ *where* W *is a Gaussian random variable (not necessarily with mean* 0*).*

The most familiar instance of this situation is *multifractional Brownian motion*

$$Y(t) = \int_{-\infty}^{\infty} \left((t-x)_+^{h(t)-1/2} - (-x)_+^{h(t)-1/2}\right) W(dx) \quad (t \in \mathbb{R}), \tag{2.4}$$

where $h : \mathbb{R} \to (0,1)$ is a Lipschitz function (a condition that may be weakened). Then $Y'_u = B_{h(u)}$ is index-$h(u)$ fractional Brownian motion, see [**AA**, **AL2**, **BJR**, **EM**, **PL**].