Séminaires & Congrès

COLLECTION SMF

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Mar Yor

SELF-SIMILAR PROCESSES AND THEIR APPLICATIONS

Numéro 28

July 20-24, 2009. Angers, France

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
28, 2012, p. 115–121

ON WEAK AND STRONG BROWNIAN FILTRATIONS: DEFINITIONS AND EXAMPLES

by

Mar Yor

Abstract. — This short note consists of 3 parts: [A], [B], [C], which are devoted respectively to:
- the presentation of the notions of weak and strong Brownian filtrations;
- the fact that the perturbation of the Brownian filtration under an absolutely continuous change of measure is always weakly Brownian, and sometimes, strictly so;
- the striking result by B. Tsirel'son that the filtration of the Brownian spider with $N(\geq 3)$ legs is strictly weakly Brownian.

Résumé **(Sur les filtrations browniennes faibles et fortes : définitions et exemples)**
Cet note est composée de trois parties, [A], [B] et [C], qui concernent respectivement :
- la présentation des notions de filtrations browniennes faibles et fortes ;
- le fait que la perturbation d'une filtration brownienne par un changement de mesure absolument continu est toujours faiblement brownienne, et parfois même, strictement faiblement brownienne ;
- le résultat remarquable de B. Tsirel'son selon lequel la filtration de l'araignée brownienne, comportant $N(\geq 3)$ pattes est strictement faiblement brownienne.

The character of this note is of concise survey type. No new result is presented.
I should also add that Michel Emery has published several important papers on this subject, in particular linking the above weak-strong discussion with cosiness and standardness properties of filtrations, and works by A. Vershik.
Some of Michel Emery's papers are of survey type, which allows me not to get deeper into the subject, this note being directed primarily to readers who are newcomers to the subject.

2010 Mathematics Subject Classification. — 60J65.
Key words and phrases. — weak, strong, Brownian filtrations, perturbation, Brownian spider.

[A] Paul-André Meyer, and more generally, the Strasbourg school of probability, have put a lot of emphasis on the study of filtrations $(\mathcal{F}_t)_{t\geq 0}$ within a probability space $(\Omega, \mathcal{F}, P)$.
This emphasis may be justified by the importance played by martingales $(M_t)_{t\geq 0}$ in modern probability theory. However, the martingale property for a random process $(M_t)_{t\geq 0}$ is relative to a filtration $(\mathcal{F}_t)$:

$$E[M_t|\mathcal{F}_s] = M_s\ ,\ s \leq t \tag{1}$$

and we should say, if (1) is satisfied that (M_t) is a martingale with respect to $(\mathcal{F}_t)$; if $(\mathcal{F}_t)$ is replaced by $(\mathcal{F}'_t)$ with: $\mathcal{F}_t \subset \mathcal{F}'_t$, it is not true in general that, for $s \leq t$, $E[M_t|\mathcal{F}'_s]$ shall still be equal to M_s.
Going one step further, one may ask, when trying to make some parallel between random processes and filtrations, to describe, or even better, to characterize Brownian filtrations that is filtrations $(\mathcal{F}_t)$ which are generated precisely by some one-dimensional Brownian motion $(B_t, t \geq 0)$. (Of course, one might ask the question with d-dimensional Brownian motion, for some $d \in \mathbb{N}$, but let us start with $d = 1$!).
A well-known, and fundamental, theorem due to K. Itô, asserts that if $\mathcal{B}_t \equiv \sigma\{B_s, s \leq t\}$, $t \geq 0$, is the natural filtration of B, then every (local) martingale $(M_t, t \geq 0)$ with respect to $(\mathcal{B}_t)$ may be written as:

$$M_t = c + \int_0^t m_s dB_s \tag{2}$$

for some constant $c \in \mathbb{R}$, and some $(\mathcal{B}_s)$-previsible process $(m_s, s \geq 0)$ such that: $\int_0^t m_s^2 ds$ a.s, for every $t < \infty$. In particular, every $(\mathcal{B}_t)$ martingale admits a continuous version, but (2) is much more precise...
We also note that some modification of (2) may be obtained by considering any $(\mathcal{B}_s)$-previsible process $(\varepsilon_s, s \geq 0)$ taking values in $\{-1, +1\}$, and writing (2) in the form:

$$M_t = c + \int_0^t m_s^{(\varepsilon)} dB_s^{(\varepsilon)} \tag{$(2)_\varepsilon$}$$

where: $B_t^{(\varepsilon)} = \int_0^t \varepsilon_s dB_s$, $t \geq 0$, is another $(\mathcal{B}_t)$-Brownian motion, and $m_s^{(\varepsilon)} = m_s \varepsilon_s$.
Note that, for some processes (ε_s) as above, the filtration $\mathcal{B}_t^{(\varepsilon)} \equiv \sigma(B_s^{(\varepsilon)}, s \leq t)$ may be strictly contained in $(\mathcal{B}_t)$, e.g: this is the case when $\varepsilon_s = \mathrm{sgn}(B_s)$, since then $\mathcal{B}_t^{(\varepsilon)} = \sigma\{|B_s|, s \leq t\}$ but, nonetheless, $(2)_\varepsilon$ holds.
The above discussion leads us naturally to the notion of a *weak Brownian filtration*

$(\mathcal{F}_t)$, that is a filtration with respect to which there exists a Brownian motion $(\beta_t, t \geq 0)$ such that every $(\mathcal{F}_t)$ martingale may be written as:

$$M_t = c + \int_0^t m_s d\beta_s \tag{3}$$

for some $(\mathcal{F}_s)$ predictable process (m_s), and $c(\equiv E(M_t)) \in \mathbb{R}$.

In [1], I conjectured - very naïvely indeed! - that a weak Brownian filtration is always a (strong) Brownian filtration, i.e: that there should exist a Brownian motion $(B_t, t \geq 0)$, such that: $\mathcal{F}_t = \sigma\{B_s, s \leq t\}$. It took some time before this conjecture was proven to be wrong, but nowadays, thanks to - in particular - fundamental imports and results by Tsirel'son, and co-authors, and further authors... (see below), many examples of weak Brownian filtrations which are not strong ones have been given, sometimes in a very explicit manner. In the next two paragraphs, I shall discuss two families of weak (sometimes strong) Brownian filtrations:

- the first family is obtained when changing the Wiener measure on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ in an absolutely continuous manner; this discussion is a kind of parallel to the discussion following (1), in which the filtration was being changed. Here, it is the probability P which is modified;
- the second discussion is about the filtration of the Brownian spider $(\mathcal{S}_t^{(N)}, t \geq 0)$ whose state space is the union of $N \geq 3$ half-lines emanating from a point. It was shown by Tsirel'son ([2]) and much discussed later, that, for $N \geq 3$, the filtration $\mathcal{T}_t^{(N)} = \sigma\{\mathcal{S}_s^{(N)}; s \leq t\}$, $t \geq 0$, is weakly, but not strongly, Brownian.

[B] The canonical filtration $\mathcal{F}_t \equiv \sigma\{X_s, s \leq t\}$ under Wiener measure W, perturbed by an absolutely continuous change of probability, may be (only) weakly Brownian.

(B.1) In this section, we consider a strictly positive martingale $(D_t, t \geq 0)$ with respect to $(W, \mathcal{F}_t)$ on the canonical space $\Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, where $X_t(\omega) = \omega(t)$ denotes the process of coordinates.

We then consider the new probability $\widetilde{W}$, which satisfies:

$$\widetilde{W}_{|\mathcal{F}_t} = D_t \bullet W_{|\mathcal{F}_t}$$

(we assume that: $E_W(D_t) \equiv D_0 = 1$).

(B.2) We first prove that, under $\widetilde{W}$, the filtration $(\mathcal{F}_t)$ is weakly Brownian.

Girsanov's theorem tells us that, for any $(W, (\mathcal{F}_t))$ (local) martingale (M_t), the process:

$$\mathcal{G}(M)_t = M_t - \int_0^t \frac{d < M, D >_s}{D_s}\,,\ t \geq 0\,,$$

is a (local) $(\widetilde{W}, (\mathcal{F}_t))$ martingale. In particular, $(\mathcal{G}(X)_t, t \geq 0)$ is a $(\widetilde{W}, (\mathcal{F}_t))$ Brownian motion, and we shall show below that every $(\widetilde{W}, (\mathcal{F}_t))$ martingale $(\widetilde{M}_t, t \geq 0)$ may be written as:

$$\widetilde{M}_t = c + \int_0^t \widetilde{m}_s d(\mathcal{G}(X)_s) \tag{4}$$

Hence, under $\widetilde{W}$, $(\mathcal{F}_t)$ is a weak Brownian filtration.

Proof of (4): $(\widetilde{M}_t)$ is a $(\widetilde{W}, (\mathcal{F}_t))$ local martingale iff $\widetilde{M}_t D_t \stackrel{\text{def}}{=} M_t$ is a $(W, (\mathcal{F}_t))$ local martingale. Thus, let us use Itô's formula to represent $\widetilde{M}_t = \dfrac{1}{D_t} M_t$, and finally obtain (4). We get:

$$\widetilde{M}_t = M_0 + \int_0^t \frac{dM_s}{D_s} + \int_0^t M_s d\left(\frac{1}{D_s}\right) + < M, \frac{1}{D} >_t \tag{5}$$

We also have:

$$\begin{aligned} d\left(\frac{1}{D_s}\right) &= -\frac{dD_s}{D_s^2} + \frac{d < D, D >_s}{D_s^3}\,, \\ &\equiv -\frac{1}{D_s^2} d(\mathcal{G}(D)_s)\,, \end{aligned}$$

which, plugged in (5), yields:

$$\begin{aligned} d\widetilde{M}_s &= \frac{1}{D_s} dM_s - \frac{M_s}{D_s^2} d(\mathcal{G}(D)_s) - \int_0^t \frac{1}{D_s^2} d < M, D >_s \\ &= \frac{1}{D_s} d(\mathcal{G}(M)_s) - \frac{M_s}{D_s^2} d(\mathcal{G}(D)_s)\,. \end{aligned} \tag{6}$$

since $dM_s = m_s dX_s$, and $dD_s = \delta_s dX_s$, for some predictable processes m and δ, and since $\mathcal{G}$ commutes with stochastic integration, we get, from (6):

$$d\widetilde{M}_s = \left(\frac{m_s}{D_s} - \frac{M_s}{D_s^2}\delta_s\right) d(\mathcal{G}(X)_s) \tag{7}$$

i.e: (4) is satisfied, with the interesting formula:

$$\widetilde{m}_s = \frac{m_s}{D_s} - \frac{M_s}{D_s^2}\delta_s \tag{8}$$