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SELF-SIMILAR PROCESSES AND THEIR APPLICATIONS

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A UNIFIED FRAMEWORK FOR THE STUDY OF THE 2-MICROLOCAL AND LARGE DEVIATION MULTIFRACTAL SPECTRA

by

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Abstract. — The large deviation multifractal spectrum is a function of central importance in multifractal analysis. It allows a fine description of the distribution of the singularities of a function over a given domain. The 2-microlocal spectrum, on the other hand, provides an extremely precise picture of the regularity of a distribution at a point. These two spectra display a number of similarities: their definitions use the same kind of ingredients; both functions are semi-continuous; the Legendre transform of the two spectra yields a function of independent interest: the 2-microlocal frontier in 2-microlocal analysis, and the "τ" function in multifractal analysis. This paper investigates further these similarities by providing a common framework for the definition and study of the spectra. As an application, we obtain slightly generalized versions of the 2-microlocal and weak multifractal formalisms (with simpler proofs), as well as results on the inverse problems for both spectra.

Résumé **(Un cadre commun pour l'étude des spectres 2-microlocal et multifractal)**

Le spectre de grandes déviations est un outil d'importance centrale en analyse multifractale. Il permet une description fine de la répartition des singularités d'une fonction. Le spectre 2-microlocal, quant à lui, fournit des renseignements extrêmement fins sur la régularité d'une distribution en un point. Ces deux spectres possèdent une certain nombre de caratéristiques communes : leur définition utilise les mêmes types d'ingrédients ; les deux spectres sont des fonctions semi-continues ; enfin, la transformée de Legendre des deux spectres conduit à des fonctions qui présentent leur intérêt propre : la frontière 2-microlocale en analyse 2-microlocale, et la fonction "τ" en analyse multifractale. Ce travail étudie et prolonge ces similarités en fournissant une cadre d'étude commun pour l'analyse des deux spectres. Comme application, nous obtenons des versions un peu plus générales que celles connues dans la littérature concernant les formalismes multifractal et 2-microlocal (avec des preuves plus simples), ainsi que des résultats sur les problèmes inverses pour les deux spectres.

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1. Introduction and Background

The analysis of global regularity, classically based on the global Hölder exponent, is adapted for the study of homogeneous signals. However, the global Hölder exponent yields insufficient information when the regularity of a function evolves in "time". Studying such functions requires tools that allow to characterize their behaviour at or around any point. One such tool is the pointwise Hölder exponent. We shall denote $\alpha_p(f, x_0)$ the exponent of the function f at the point x_0. Multifractal analysis [**FP85**, **AP96**, **BMP92**, **CLP87**, **EM92**, **Fan97**, **HJK+86**, **KP76**, **Jaf97a**, **VT04**, **VV98**, **Man74**, **Ols95**] studies the structure of the pointwise Hölder function, *i.e.* the function $x_0 \to \alpha_p(f, x_0)$: more precisely it aims at obtaining the *multifractal spectrum*, a function which measures the "size" of the level lines of $\alpha_p(f, x)$. Both the theoretical and the numerical computations of this spectrum are difficult. This is why physicists and mathematicians have investigated a "multifractal formalism", which allows, in certain situations, to obtain the spectrum as the Legendre transform of a function that can be computed more easily.

Instead of focusing on the pointwise Hölder exponent and the fine structure of $\alpha_p(f, x_0)$, one may follow a different approach and try to obtain a richer description of the local regularity at any fixed point by means of other exponents, such as the local Hölder exponent [**GL98**], the chirp exponent [**Mey98**], the oscillation exponent [**ABJM98**] or the "weak scaling" exponent [**Mey98**]. A powerful way to do so is to study the 2-microlocal frontier, defined in [**GL98**, **Mey98**] based on the local version of the 2-microlocal spaces introduced by J.M. Bony in [**Bon83**]. The main interest of these spaces is that they allow to describe completely the evolution of the pointwise Hölder exponent at any given point under integro-differentiation. The 2-microlocal frontier is a curve in an abstract space that is associated to each point, and that allows to predict this evolution. The 2-microlocal spaces were originally defined through a Littlewood-Paley decomposition. They were then characterized by conditions on the wavelet coefficients [**Jaf91**]. Time domain characterizations of increasing generality have been provided in [**KL02**, **LS04**, **Ech07**]. See also [**Mey98**] for related results.

The computation of the 2-microlocal frontier is somewhat delicate. A 2-microlocal formalism has been studied in [**GL98**, **LS04**, **Ech07**], with an approach that is analogous, in many respects, to the one of the multifractal formalism: at any fixed point, the 2-microlocal frontier is the Legendre transform of a certain function called the 2-microlocal spectrum.

Thus, for a function f, the multifractal spectrum characterizes the level sets of the pointwise Hölder function, while the 2-microlocal spectrum allows to predict the change of regularity by integro-differentiation at any point in the domain of f.

These two descriptions yield a rather rich picture of the regularity, and they may be approached through related formalisms, which are essentially based on a Legendre transform.

In this work, we elaborate on the similitudes between the two formalisms. We also study the problem of prescribing both the 2-microlocal and multifractal spectra. In the next section, we expose some general notions that are useful in both settings. In Section 3, we provide an abstract (weak) formalism. This formalism is applied to various versions of the multifractal spectra in Section 4, and to the 2-microlocal spectrum in Section 5. Finally, Section 6 presents results on the prescription of the spectra.

Sections 2 and 3 stay at a very general level. As a consequence, the definitions and results they present might appear rather abstract to the reader. However, as will be apparent in Sections 4 and 5, they contain the essence of what is common to the multifractal and 2-microlocal formalisms. In particular, propositions 4.1, 4.8, 5.2 and 5.3 may be seen as concrete examples of applications of this abstract formalism. In fact, the results in Sections 2 and 3 elucidate the very mechanisms relating the spectra, and might have applications in other settings.

2. Recalls: basic properties of functions defined on $\mathcal{P}(\mathbb{R})$

We recall in subsections 2.1, 2.2 and 2.3 some known definitions and results on set functions. In subsection 2.4, we specialize to a case which will be relevant for both 2-microlocal and multifractal analysis.

2.1. Common frame. — We consider a function $F : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. F is called *non-decreasing* if for all real sets E_1, E_2,

$$E_1 \subset E_2 \Rightarrow F(E_1) \leq F(E_2).$$

It is called *stable* (with respect to union) if

$$F(E_1 \cup E_2) = \max\{F(E_1), F(E_2)\}.$$

Given a real k, F is *k-stable* if

$$F(E_1 \cup E_2) - k \leq \max\{F(E_1), F(E_2)\} \leq F(E_1 \cup E_2) + k.$$

It is easily checked that any function $G : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}$ that is non-decreasing and sub-additive (*i.e.* $G(E_1 \cup E_2) \leq G(E_1) + G(E_2)$) is such that $F := \log G$ is k-stable with $k = \log 2$.

2.2. Non-decreasing F. — Assume that F is non-decreasing. For all real α, the function $F([\alpha-\epsilon,\alpha+\epsilon])$ has a limit when ϵ tends to 0. One may then define the *localized function* (also called *max-plus limit*).

$$F^{loc}(\alpha) = \lim_{\epsilon\to 0} F([\alpha-\epsilon,\alpha+\epsilon]).$$

Lemma 2.1. — *Let F be non-decreasing. For all open set O in $\mathbb{R}$,*

$$\sup_{\alpha\in O} F^{loc}(\alpha) \le F(O). \tag{2.1}$$

Proof. — Indeed, if $\alpha \in O$, there exists ϵ such that $[\alpha-\epsilon,\alpha+\epsilon] \subset O$, and thus $F([\alpha-\epsilon,\alpha+\epsilon]) \le F(O)$. □

Lemma 2.2. — *If F is non-decreasing, then F^{loc} is upper-semi-continuous.*

Proof. — The preceding lemma implies that, for $\epsilon > 0$, $\sup_{\beta\in(\alpha-\epsilon,\alpha+\epsilon)} F^{loc}(\beta) \le F((\alpha-\epsilon,\alpha+\epsilon))$. Letting ϵ tend to 0, one gets the semi-continuity of F^{loc} in α. □

Lemma 2.3. — *If F is non-decreasing, then F^{loc} reaches its supremum on any compact set of $\mathbb{R}$.*

Proof. — This is a direct consequence of the semi-continuity. □

2.3. Stability of F. — If F is stable, then it is also non-decreasing: indeed, if $E_1 \subset E_2$,

$$F(E_1) \le \max\{F(E_1), F(E_2)\} = F(E_1 \cup E_2) = F(E_2).$$

Lemma 2.4. — *Let F be stable. For any compact set K in $\mathbb{R}$,*

$$F(K) \le \max_{\alpha\in K} F^{loc}(\alpha). \tag{2.2}$$

Proof. — The proof uses the closed dyadic intervals of $\mathbb{R}$. Since K is bounded, it may be covered by a finite number of dyadic interval of rank 0 (that is, of length 1). Since F is stable, one of these intervals, denoted J_0, is such that $F(J_0\cap K) = F(K)$. However, J_0 is covered by two intervals of rank 1, say J' and J''. One of these intervals, denoted J_1, is also such that $F(J_1 \cap K) = F(K)$. By recurrence, one may construct a nested sequence (J_n) of dyadic intervals such that, for all n, $F(J_n \cap K) = F(K)$.

Let α_* denote the limit of the J_n. Since K is closed, α_* is a point of K. For any $\epsilon > 0$, there exists an integer n such that $J_n \subset [\alpha_*-\epsilon,\alpha_*+\epsilon]$. Using the fact that F is non-decreasing, one gets that: $F(J_n) \le F([\alpha_*-\epsilon,\alpha_*+\epsilon])$. Thus $F(K) \le F([\alpha_*-\epsilon,\alpha_*+\epsilon])$, and, letting ϵ tend to 0: $F(K) \le F^{loc}(\alpha_*)$. As a consequence, $F(K) \le \sup_{\alpha\in K} F^{loc}(\alpha)$.

Finally, since F is non-decreasing, F^{loc} reaches its supremum on K. □