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SELF-SIMILAR PROCESSES AND THEIR APPLICATIONS

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Abstract. — Selfdecomposable distributions, stationary Ornstein-Uhlenbeck type processes and selfsimilar additive processes are closely related and have been studied deeply. In this paper, generalizations of these concepts are introduced and investigated, which are α-selfdecomposable distributions, mild Ornstein-Uhlenbeck type processes and quasi-selfsimilar additive processes.

***Résumé* (Lois α-auto-décomposables, processus de type Ornstein-Uhlenbeck faible et processus quasi auto-similaires additifs)**

Les lois auto-décomposables, les processus stationnaires de type Ornstein-Uhlenbeck faible et les processus quasi auto-similaires additifs sont étroitement liés et ont été étudiés de manières intensive. Dans cet article, certaines généralisations de ces concepts sont proposées et étudiées. Il s'agit des lois α-auto-décomposables, des processus de type Ornstein-Uhlenbeck faible et des processus quasi auto-similaires additifs.

1. Introduction

There are close relations among selfdecomposable distributions, stationary Ornstein-Uhlenbeck type processes and selfsimilar additive processes. Let $\mathscr{P}(\mathbb{R}^d)$ and $I(\mathbb{R}^d)$ be the class of all probability distributions on $\mathbb{R}^d$ and the class of all infinitely divisible distributions on $\mathbb{R}^d$, respectively, and let $I_{\log}(\mathbb{R}^d)$ be the totality of $\mu \in I(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \log^+ |x|\mu(dx) < \infty$, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^d$ and $\log^+ |x| = (\log |x|) \vee 0$. A distribution $\mu \in \mathscr{P}(\mathbb{R}^d)$ is said to be selfdecomposable if for each $b > 1$ there exists $\rho_b \in \mathscr{P}(\mathbb{R}^d)$ satisfying $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z), z \in \mathbb{R}^d$, where $\widehat{\mu}(z), z \in \mathbb{R}^d$, stands for the characteristic function of $\mu \in \mathscr{P}(\mathbb{R}^d)$. Here μ

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and ρ_b automatically belong to $I(\mathbb{R}^d)$. We denote the totality of selfdecomposable distributions on $\mathbb{R}^d$ by $L(\mathbb{R}^d)$. Thus, $L(\mathbb{R}^d) \subset I(\mathbb{R}^d)$. A stochastic process $\{Z_t, t \in \mathbb{R}\}$ on $\mathbb{R}^d$ is said to be a stationary Ornstein-Uhlenbeck type process (OU type process) if

$$Z_t = e^{-Ht} \int_{-\infty}^{t} e^{Hu} X(du), \quad t \in \mathbb{R}, \tag{1.1}$$

where $H > 0$ and X is an $\mathbb{R}^d$-valued homogeneous independently scattered random measure (homogeneous i.s.r.m.) over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) \in I_{\log}(\mathbb{R}^d)$. Here and in what follows, $\mathcal{L}$ means "the law of". This process is known to be an almost surely unique stationary solution of the Langevin equation

$$Z_t - Z_s = \int_s^t X(du) - H \int_s^t Z_u du, \quad -\infty < s \le t < \infty. \tag{1.2}$$

Furthermore, for every $H > 0$, a stochastic process $\{Y_t, t \ge 0\}$ on $\mathbb{R}^d$ is said to be H-selfsimilar if for any $a > 0$,

$$\{Y_{at}, t \ge 0\} \stackrel{\mathrm{d}}{=} \{a^H Y_t, t \ge 0\},$$

where $\stackrel{\mathrm{d}}{=}$ stands for equality in all finite-dimensional distributions. Also, a stochastic process $\{Y_t, t \ge 0\}$ on $\mathbb{R}^d$ is called an additive process if it is a stochastically continuous càdlàg process with independent increments satisfying $Y_0 = 0$ a.s. Then, the three concepts above are related as follows, (see, e.g., [8, 10, 11]). Stationary OU type processes $\{Z_t\}$ satisfy $\mathcal{L}(Z_t) = \mathcal{L}\left(\int_0^\infty e^{-Hu} X(du)\right) \in L(\mathbb{R}^d)$ for all $t \in \mathbb{R}$. Conversely, for any $\mu \in L(\mathbb{R}^d)$, there is a stationary OU type process $\{Z_t\}$ on $\mathbb{R}^d$ satisfying $\mu = \mathcal{L}(Z_t)$ for all $t \in \mathbb{R}$. Also, stationary OU type processes correspond to selfsimilar additive processes through the Lamperti transformation introduced in [6]. Namely, if $\{Z_t, t \in \mathbb{R}\}$ is a stationary OU type process (1.1), then $\{Y_t, t \ge 0\}$ defined by

$$Y_t = \begin{cases} t^H Z_{\log t}, & \text{for } t > 0, \\ 0, & \text{for } t = 0, \end{cases}$$

is a H-selfsimilar additive process on $\mathbb{R}^d$. Conversely, if $\{Y_t, t \ge 0\}$ is a selfsimilar additive process on $\mathbb{R}^d$, then $\{Z_t, t \in \mathbb{R}\}$, defined by $Z_t = e^{-Ht} Y_{e^t}$, is a stationary OU type process. Moreover, the following relations between selfsimilar additive processes and selfdecomposable distributions are known. Any marginal distribution of a selfsimilar additive process is selfdecomposable. Conversely for any $\mu \in L(\mathbb{R}^d)$ there is a selfsimilar additive process with μ as its distribution at time 1.

The purpose of this paper is to introduce and study generalizations of these three concepts. Selfdecomposable distributions and stationary OU type processes have already been generalized to α-selfdecomposable distributions and mild OU type processes, respectively, in [9] as in the following two definitions.

Definition 1.1. — Let $\alpha \in \mathbb{R}$. We say that $\mu \in I(\mathbb{R}^d)$ is *α-selfdecomposable*, if for any $b > 1$, there exists $\rho_b \in I(\mathbb{R}^d)$ satisfying

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)^{b^\alpha} \widehat{\rho}_b(z), \quad z \in \mathbb{R}^d. \tag{1.3}$$

We denote the totality of α-selfdecomposable distributions on $\mathbb{R}^d$ by $L^{\langle \alpha \rangle}(\mathbb{R}^d)$.

Throughout this paper, when we consider the case $\alpha = 1$, we will need the following function q_μ. For $\mu = \mu_{(A,\nu,\gamma)} \in I(\mathbb{R}^d)$, define a nonrandom continuous function $q_\mu \colon (-\infty, 1) \to \mathbb{R}^d$ by

$$q_\mu(t) := \begin{cases} \displaystyle\int_t^0 (1-u)^{-1} du \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+(1-u)^{-2}|x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right), & t \le 0, \\ 0, & 0 < t < 1. \end{cases}$$

If a random variable M satisfies $\mathcal{L}(M) = \mu \in I(\mathbb{R}^d)$, then we may also write q_M for q_μ.

Let

$$I_\alpha(\mathbb{R}^d) = \left\{ \mu \in I(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \right\}, \quad \text{for } 0 < \alpha < 2,$$

$$I_\alpha^0(\mathbb{R}^d) = \left\{ \mu \in I_\alpha(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} x \mu(dx) = 0 \right\}, \quad \text{for } 1 \le \alpha < 2,$$

$$I_1^*(\mathbb{R}^d) = \left\{ \mu_{(A,\nu,\gamma)} \in I_1^0(\mathbb{R}^d) \colon \lim_{T \to \infty} \int_1^T t^{-1} dt \int_{|x|>t} x \nu(dx) \text{ exists in } \mathbb{R}^d \right\}.$$

Definition 1.2. — Suppose that X is an $\mathbb{R}^d$-valued homogeneous i.s.r.m. over $\mathbb{R}$ with $\mathcal{L}(X((0,1])) = \mu$.

(i) Let $\alpha < 0$. Then, a stochastic process $\{Z_t\}$ on $\mathbb{R}^d$ is said to be a *mild OU type process generated by* (α, X) if

$$Z_t = (1-\alpha t)^{1/\alpha} \int_{1/\alpha}^t (1-\alpha u)^{-1/\alpha} X(du), \quad 1/\alpha < t < \infty. \tag{1.4}$$

(ii) Let $0 < \alpha < 1$ and $\mu \in I_\alpha(\mathbb{R}^d)$. Suppose that S_α is an $\mathbb{R}^d$-valued strictly α-stable random variable independent of X. Then, a stochastic process $\{Z_t\}$ on $\mathbb{R}^d$ is said to be a *mild OU type process generated by* (α, X) *associated with* S_α if

$$Z_t = (1-\alpha t)^{1/\alpha} \left\{ S_\alpha + \int_{-\infty}^t (1-\alpha u)^{-1/\alpha} X(du) \right\}, \quad -\infty < t < 1/\alpha. \tag{1.5}$$

(iii) Let $\alpha = 1$ and $\mu \in I_1(\mathbb{R}^d)$. Suppose that S_1 is an $\mathbb{R}^d$-valued 1-stable random variable independent of X. Then, a stochastic process $\{Z_t\}$ on $\mathbb{R}^d$ is said to be a *mild OU type process generated by* $(1, X)$ *associated with* S_1 if

$$Z_t = (1-t)\left\{S_1 + \operatorname*{p\text{-}lim}_{s \downarrow -\infty}\left(\int_s^t (1-u)^{-1} X(du) - q_\mu(s)\right)\right\}, \quad -\infty < t < 1, \tag{1.6}$$

where p-lim means limit in probability.

(iv) Let $1 < \alpha < 2$ and $\mu \in I_\alpha^0(\mathbb{R}^d)$. Suppose that S_α is an $\mathbb{R}^d$-valued α-stable random variable independent of X. Then, a stochastic process $\{Z_t\}$ on $\mathbb{R}^d$ is said to be a *mild OU type process generated by* (α, X) *associated with* S_α if $\{Z_t\}$ has the same expression as that in (1.5).

When $\alpha < 0$, we also call $\{Z_t\}$ just an *α-mild OU type process* if $\{Z_t\}$ is a mild OU type process generated by (α, X) for some X. Similarly, when $\alpha > 0$, we call $\{Z_t\}$ an *α-mild OU type process associated with* S_α if $\{Z_t\}$ is a mild OU type process generated by (α, X) associated with S_α for some X which is independent of S_α and fulfills the moment condition above. This X is called a *background driving homogeneous i.s.r.m.* of $\{Z_t\}$.

We now introduce the concept of quasi-selfsimilarity of additive processes.

Definition 1.3. — Let $\alpha \in \mathbb{R}$ and $\{Y_t, t \geq 0\}$ be an additive processes on $\mathbb{R}^d$. We call $\{Y_t\}$ a *broad-sense α-quasi-selfsimilar additive process* if for any $a > 0$, there exists a function $c_a(t)$ such that for each $t \geq 0$,

$$\mathcal{L}(Y_{at}) = \mathcal{L}\left(aY_t + c_a(t)\right)^{a^{-\alpha}}, \tag{1.7}$$

where $\mathcal{L}\left(aY_t + c_a(t)\right)^{a^{-\alpha}}$ is the distribution whose characteristic function is the $a^{-\alpha}$-th power of that of $\mathcal{L}\left(aY_t + c_a(t)\right)$. If we can take $c_a(t) \equiv 0$ for all $a > 0$, then we call $\{Y_t\}$ an *α-quasi-selfsimilar additive process.*

Note that this concept depends only on equality of distributions of two processes for each fixed time t, which is different from the definition of ordinary selfsimilarity.

As to the generalized concepts above, we give several remarks as follows.

Remark 1.4. — **(1)** We have $L^{\langle 0 \rangle}(\mathbb{R}^d) = L(\mathbb{R}^d)$. Also, $L^{\langle -1 \rangle}(\mathbb{R}^d)$ is the class of all s-selfdecomposable distributions on $\mathbb{R}^d$, which is sometimes written as $U(\mathbb{R}^d)$ and was studied deeply by Jurek, (see, e.g., [1, 3, 4, 5]). Also, the classes $L^{\langle \alpha \rangle}(\mathbb{R}^d), \alpha \in \mathbb{R}$, and similar ones were already studied by several authors. For details on this history, see [9].